ILL-POSEDNESS FOR SUBCRITICAL HYPERDISSIPATIVE NAVIER-STOKES EQUATIONS IN THE LARGEST CRITICAL SPACES

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ABSTRACT. We study the incompressible Navier-Stokes equations with a fractional Laplacian and prove the existence of discontinuous Leray-Hopf solutions in the largest critical space with arbitrarily small initial data.

1. INTRODUCTION

In this paper we study the supercritical 3D Navier-Stokes equations with a fractional power of the Laplacian

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = -\nu(-\Delta)^\alpha u, & x \in \mathbb{T}^3, t \geq 0, \\ \nabla \cdot u = 0, \\ u(0) = u_0, \end{cases} \tag{1}$$

where the velocity $u(x,t)$ and the pressure $p(x,t)$ are unknowns, $u_0 \in L^2(\mathbb{T}^3)$ is the initial condition, $\nu > 0$ is the kinematic viscosity coefficient of the fluid, and $\alpha > 0$. The case $\alpha = 1$ corresponds to the classical Navier-Stokes equations, which has been studied extensively for decades. We refer to [7, 17] for the classical theory for these equations. In the case $\alpha \geq 5/4$ the equations are well-posed, as the dissipative term simply dominates the nonlinear term. Moreover, the global regularity is known even in a slightly supercritical case, i.e., when logarithmic corrections to the Fourier multiplier of the dissipative term are present (see [16, 4]). However, a finite time blow up of solutions to (1) remains a possibility for $\alpha < 5/4$ due to a supercritical nature of the equations. Nevertheless, a partial regularity result [3] has been established in the supercritical case $\alpha = 1$, later extended to $\alpha \in (1, 5/4)$ in [11]. There are also various regularity criteria in the case $\alpha = 1$, most of which are of Ladyzhenskaya-Prodi-Serrin type [8, 13, 14, 15, 10, 6, 4], which can also be extended to $\alpha \in (1, 5/4)$.

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One of the open questions studied extensively is whether solutions bounded in the largest critical case ($\dot{B}^{-1}_{\infty,\infty}$ for $\alpha = 1$) are regular. A positive answer to this question would extend the famous $L^\infty_t L^3_x$ result due to Iscauriaza, Seregin, and Šverák [10]. In addition, the best small initial result for the 3D NSE, due to Koch and Tataru [12], is in the space BMO^{-1}, and it is not known either if its extension to the $B^{-1}_{\infty,\infty}$ is possible.

In view of these problems two "negative" results have been obtained in the space $\dot{B}^{-1}_{\infty,\infty}$. First, Bourgain and Pavlovic [2] proved that that there are solutions to the 3D NSE equations, with arbitrary small initial data in $\dot{B}^{-1}_{\infty,\infty}$ that become arbitrarily large in $\dot{B}^{-1}_{\infty,\infty}$ in arbitrarily small time. Second, Leray-Hopf solutions with arbitrary small initial data, but discontinuous in $B^{-1}_{\infty,\infty}$ were obtained in [5].

The largest critical space for the fractional NSE (1) is $\dot{B}^{1-2\alpha}_{\infty,\infty}$. Recently Yu and Zhai [18] obtained a small initial data result in this space in the hypodissipative case $\alpha \in (1/2, 1)$. Heuristically, the hypodissipative NSE behaves better because it is closer to the fractional heat semigroup in critical spaces. In the hyperdissipative case it is therefore natural to expect ill-posedness results of the type mentioned above. Indeed, in this paper we demonstrate this in the case $\alpha \in [1, 5/4)$ by constructing a Leray-Hopf solution with arbitrarily small initial data, which is discontinuous in the critical Besov space $B^{1-2\alpha}_{\infty,\infty}$. It is thus a direct extension of our previous result stated in [5]. The method breaks down either when α passes beyond the value of 1, which is consistent with the result of Yu and Zhai, and at $5/4$ and beyond, which is consistent with the global regularity in that range.

We now fix our notation. We assume periodic boundary conditions in all 3 dimensions, so $\mathbb{T}^3$ will denote the 3D torus, while $|\cdot|_p$, $p \geq 1$, denotes the L^p-norm in $\mathbb{T}^3$. We let $\hat{f}$ and $\check{f}$ stand for the forward and, respectively, inverse Fourier transforms on the torus. The Fourier multiplier with symbol $|\xi|^\alpha$, where ξ stands for the frequency vector and $\alpha > 0$, is denoted by $|\nabla|^\alpha$. The fractional Laplacian operator $(-\Delta)^\alpha$ has symbol $|\xi|^{2\alpha}$. We write $p(\xi) = \mathrm{id} - |\xi|^{-2}\xi \otimes \xi$, $\xi \neq 0$, $p(0) = \mathrm{id}$, for the symbol of the Leray-Hopf projection on the divergence-free fields. We fix notation for the dyadic a-dimensional wavenumbers $\lambda_q = 2^q$. We use extensively the classical dyadic decomposition throughout: $u = \sum_{q\geq 0} u_q$, where u_q is the Littlewood-Paley projection with the Fourier support contained in $\{\lambda_{q-1} < |\xi| < \lambda_{q+1}\}$. The definitions are standard and can be found in the references above. We often will be using the extended projection defined by $\tilde{u}_q = u_{q-1} + u_q + u_{q+1}$, $q \geq 1$, and projection onto the dyadic ball, $u_{\leq q} = \sum_{p=0}^q u_p$. Thus, $\tilde{u}_q$ is supported on $\{\lambda_{q-2} < |\xi| < \lambda_{q+2}\}$ and we have the identity

$$\int_{\mathbb{T}^3} u \cdot u_q \, dx = \int_{\mathbb{T}^3} \tilde{u}_q \cdot u_q \, dx. \tag{2}$$

With the Littlewood-Paley decomposition we define Besov spaces $B^s_{r,\infty}$, $s \in \mathbb{R}$, $r \geq 1$ by requiring

$$\|u\|_{B^s_{r,\infty}} = \sup_{q \geq 0} \lambda_q^s \|u_q\|_r < \infty.$$

We will frequently refer to Bernstein's inequalities, which state that for all $1 \leq r < r' \leq \infty$, and in three dimensions, one has

$$|u_q|_{r'} \lesssim \lambda_q^{3(1/r - 1/r')} |u_q|_r,$$

where here and throughout $\lesssim$ denote inequality up to an absolute constant. Finally, let $\vec{e}_1$, $\vec{e}_2$, etc., stand for the vectors of the standard unit basis.

2. Ill-posedness of NSE

The Navier-Stokes equation with a fractional power of the Laplacian is given by

$$u_t + (u \cdot \nabla) u = -\nu (-\Delta)^\alpha u - \nabla p. \tag{3}$$

Here u is a three dimensional divergence free field on $\mathbb{T}^3$, and $\alpha \in [1, 5/4)$. Let us recall that for every field $U \in L^2(\mathbb{T}^3)$ there exists a weak solution $u \in C_w([0,T); L^2) \cap L^2([0,T); H^1)$ to (3) such that the energy inequality

$$|u(t)|_2^2 + 2\nu \int_0^t ||\nabla|^\alpha u(s)|_2^2 ds \leq |U|_2^2, \tag{4}$$

holds for all $t > 0$ and $u(t) \to U$ strongly in L^2 as $t \to 0$. In what follows we do not actually use inequality (4) which allows us to formulate a more general statement below in Proposition 2.2.

Let us choose a strictly increasing sequence $\{q_i\} \in \mathbb{N}$ with elements sufficiently far apart so that at least $\lambda_{q_i}^{2\alpha} \lambda_{q_{i+1}}^{4\alpha - 5} < 1$. We consider the following lattice blocks:

$$\begin{aligned}
A_j &= \left[\frac{9}{10}\lambda_{q_j}, \frac{11}{10}\lambda_{q_j}\right] \times \left[-\frac{1}{10}\lambda_{q_j}, \frac{1}{10}\lambda_{q_j}\right]^2 \cap \mathbb{Z}^3 \\
B_j &= \left[-\frac{1}{10}\lambda_{q_j - 1}, \frac{1}{10}\lambda_{q_j - 1}\right]^2 \times \left[\frac{9}{10}\lambda_{q_j - 1}, \frac{11}{10}\lambda_{q_j - 1}\right] \cap \mathbb{Z}^3 \\
C_j &= A_j + B_j \\
A_j^* &= -A_j, \; B_j^* = -B_j, \; C_j^* = -C_j.
\end{aligned}$$

Thus, A_j, C_j and their conjugates lie in the q_j-th shell, while B_j, B_j^* lie in the adjacent $(q_j - 1)$-th shell. The particular choice of scaling exponents

$9/10, 11/10$, etc., is unimportant as long as the blocks fit into the their respective shells. Let us denote

$$\vec{e}_1(\xi) = p(\xi)\vec{e}_1, \quad \vec{e}_2(\xi) = p(\xi)\vec{e}_2.$$

We now define the initial condition field to be the following sum

$$U = \sum_{j \geq 1} (U_{q_j} + U_{q_j - 1}), \tag{5}$$

where the components, on the Fourier side, are

$$\widehat{U_{q_j}}(\xi) = \lambda_{q_j}^{2\alpha-4} \left(\vec{e}_2(\xi)\chi_{A_j \cup A_j^*} + i(\vec{e}_2(\xi) - \vec{e}_1(\xi))\chi_{C_j} - i(\vec{e}_2(\xi) - \vec{e}_1(\xi))\chi_{C_j^*} \right),$$

and

$$\widehat{U_{q_j-1}}(\xi) = \lambda_{q_j}^{2\alpha-4} \vec{e}_1(\xi)\chi_{B_j \cup B_j^*}.$$

By construction, $\hat{U}(-\xi) = \overline{\hat{U}(\xi)}$, which ensures that U is real. Since U has no modes in the (q_j+1)-st shell, then the extended Littlewood-Paley projection of the j-th component has the form $\tilde{U}_{q_j} = U_{q_j - 1} + U_{q_j}$.

Lemma 2.1. *We have* $U \in B_{r,\infty}^{1+\frac{3}{r}-2\alpha}$*, for any* $1 < r \leq \infty$.

Proof. We give the estimate only for one block, the other ones being similar. Using boundedness of the Leray-Hopf projection, we have, for all $1 < r < \infty$,

$$|\lambda_{q_j}^{2\alpha-4}(\vec{e}_2(\cdot)\chi_{A_j})^\vee|_r \lesssim \lambda_{q_j}^{2\alpha-4}|(\chi_{A_j})^\vee|_r.$$

Notice that by construction,

$$|(\chi_{A_j})^\vee(x_1, x_2, x_3)| = |D_{(c+1)\lambda_{q_j}}(x_1) D_{c\lambda_{q_j}}(x_2) D_{c\lambda_{q_j}}(x_3)|.$$

where D_N denotes the Dirichlet kernel. Hence,

$$|(\chi_{A_j})^\vee|_r \leq |D_{(c+1)\lambda_{q_j}}|_r |D_{c\lambda_{q_j}}|_r^2.$$

By a well-known estimate, we have $|D_N|_r \leq N^{1-\frac{1}{r}}$ (c.f. [9]). Putting the above estimates together implies the desired inclusion in $B_{r,\infty}^{1+3/r-2\alpha}$. In the case $r = \infty$ we simply use the triangle inequality to obtain

$$|U_{q_j}|_\infty \lesssim \lambda_{q_j}^{2\alpha-1}.$$

□

Let us now examine the trilinear term. We will use the following notation for convenience

$$u \otimes v : \nabla w = \int_{\mathbb{T}^3} v_i \partial_i w_j u_j dx. \tag{6}$$

Using the antisymmetry we obtain

$$\begin{aligned}
U \otimes U : \nabla U_{q_j} &= \sum_{k \geq j+1} \tilde{U}_{q_k} \otimes \tilde{U}_{q_k} : \nabla U_{q_j} + \tilde{U}_{q_j} \otimes \tilde{U}_{q_j} : \nabla U_{q_j} \\
&+ U_{\leq q_{j-1}} \otimes \tilde{U}_{q_j} : \nabla U_{q_j} + \tilde{U}_{q_j} \otimes U_{\leq q_{j-1}} : \nabla U_{q_j} \\
&= \sum_{k \geq j+1} \tilde{U}_{q_k} \otimes \tilde{U}_{q_k} : \nabla U_{q_j} + U_{q_j-1} \otimes U_{q_j} : \nabla U_{q_j} \\
&- U_{q_j} \otimes U_{q_j} : \nabla U_{\leq q_{j-1}} \\
&= A + B + C.
\end{aligned}$$

Using Bernstein's inequalities we estimate

$$\begin{aligned}
|A| &\lesssim \lambda_{q_j} |U_{q_j}|_\infty \sum_{k \geq j+1} |\tilde{U}_{q_k}|_2^2 \lesssim \lambda_{q_j}^{2\alpha} \lambda_{q_{j+1}}^{4\alpha-5} \leq 1, \\
|C| &\lesssim |U_{q_j}|_2^2 \sum_{k \leq j-1} \lambda_{q_k} |\tilde{U}_{q_k}|_\infty \lesssim \lambda_{q_{j-1}}^{2\alpha} \lambda_{q_j}^{4\alpha-5} \leq 1,
\end{aligned}$$

where in the latter inequality we used the fact $|U_{q_j}|_2 \sim \lambda_{q_j}^{2\alpha-5/2}$. On the other hand, a straightforward computation shows that

$$B \sim \lambda_{q_j}^{6\alpha-5}, \tag{7}$$

which is thus the dominant term of the three, and hence,

$$U \otimes U : \nabla U_{q_j} \sim \lambda_{q_j}^{6\alpha-5}.$$

Proposition 2.2. *Let $u \in C_w([0,T); L^2) \cap L^2([0,T); H^1)$ be a weak solution to the NSE with initial condition $u(0) = U$. Then there is $\delta = \delta(u) > 0$ such that*

$$\limsup_{t \to 0+} \|u(t) - U\|_{B^{1-2\alpha}_{\infty,\infty}} \geq \delta. \tag{8}$$

If, in addition, u is a Leray-Hopf solution satisfying the energy inequality (4)*, then δ can be chosen independent of u.*

Proof. Let us test (3) with u_{q_j}. Using (2), we find

$$\partial_t (\tilde{u}_{q_j} \cdot u_{q_j}) = -\nu |\nabla|^\alpha \tilde{u}_{q_j} \cdot |\nabla|^\alpha u_{q_j} + u \otimes u : \nabla u_{q_j},$$

where as defined before, $\tilde{u}_{q_j} = u_{q_j-1} + u_{q_j} + u_{q_j+1}$. Denoting $E(t) = \int_0^t ||\nabla|^\alpha u|_2^2 ds$ we obtain

$$\begin{aligned}
|\tilde{u}_{q_j}(t)|_2^2 &\geq |U_{q_j}|_2^2 - \nu E(t) + c_1 \lambda_{q_j}^{6\alpha-5} t \\
&\quad - c_2 \int_0^t \left| u \otimes u : \nabla u_{q_j} - U \otimes U : \nabla U_{q_j} \right|_1 ds,
\end{aligned} \tag{9}$$

for some positive constants c_1 and c_2. We now show that if the conclusion of the proposition fails then for some small $t > 0$ the integral term the growth of the integral term above becomes less than $c_1\lambda_{q_j}^{6\alpha-5}t$ for large j. This forces $|\tilde{u}_{q_j}(t)|_2^2 \gtrsim \lambda_{q_j}^{6\alpha-5}t$ for all large j. Hence u has infinite energy, which is a contradiction.

So suppose that for every $\delta > 0$ there exists $t_0 = t_0(\delta) > 0$ such that $\|u(t) - U\|_{B^{1-2\alpha}_{\infty,\infty}} < \delta$ for all $0 < t \leq t_0$. Denoting $w = u - U$ we write

$$u \otimes u : \nabla u_{q_j} - U \otimes U : \nabla U_{q_j} = w \otimes U : \nabla U_{q_j} + u \otimes w : \nabla U_{q_j} \\ + u \otimes u : \nabla w_{q_j} = A + B + C.$$

We will now decompose each triplet into three terms according to the type of interaction (c.f. Bony [1]) and estimate each of them separately.

$$A = \sum_{\substack{p',p''\geq q_j \\ |p'-p''|\leq 2}} w_{p'} \otimes U_{p''} : \nabla U_{q_j} + w_{\leq q_j} \otimes \tilde{U}_{q_j} : \nabla U_{q_j} \\ + \tilde{w}_{q_j} \otimes U_{\leq q_j} : \nabla U_{q_j} - \text{repeated} = A_1 + A_2 + A_3.$$

Let us fix $r \in (1, 3/(4\alpha-2))$ and use Lemma 2.1 along with Hölder and Bernstein's inequalities to estimate A_1:

$$|A_1| \leq |\nabla U_{q_j}|_{r'} \sum |w_{p'}|_\infty |U_{p''}|_r \lesssim \lambda_{q_j}^{2\alpha-3+\frac{3}{r}} \sum |w_{p'}|_\infty \lambda_{p''}^{2\alpha-1-\frac{3}{r}} \\ \lesssim \delta\lambda_{q_j}^{2\alpha-3+\frac{3}{r}} \leq \delta\lambda_{q_j}^{6\alpha-5}.$$

Intergrating by parts we obtain $A_2 = U_{q_j} \otimes \tilde{U}_{q_j} : \nabla w_{\leq q_j}$. Thus, using the same tools,

$$|A_2| \leq |\tilde{U}_{q_j}|_2^2 |\nabla w_{\leq q_j}|_\infty \lesssim \lambda_{q_j}^{4\alpha-5} \sum_{p\leq q_j} \lambda_p |w_p|_\infty < \delta\lambda_{q_j}^{6\alpha-5}.$$

And finally,

$$|A_3| \leq \lambda_{q_j} |U_{\leq q_j}|_2 |U_{q_j}|_2 |\tilde{w}_{q_j}|_\infty \lesssim \lambda_{q_j}^{4\alpha-4} |\tilde{w}_{q_j}|_\infty < \delta\lambda_{q_j}^{6\alpha-5}.$$

We have shown the following estimate:

$$|A| \lesssim \delta\lambda_{q_j}^{6\alpha-5}. \tag{10}$$

As to B we decompose analogously,

$$B = \sum_{\substack{p',p''\geq q_j \\ |p'-p''|\leq 2}} u_{p'} \otimes w_{p''} : \nabla U_{q_j} + u_{\leq q_j} \otimes \tilde{w}_{q_j} : \nabla U_{q_j} \\ + \tilde{u}_{q_j} \otimes w_{\leq q_j} : \nabla U_{q_j} - \text{repeated} = B_1 + B_2 + B_3.$$

The term B is the least problematic. Here we do not even have to use the smallness of w and can just roughly estimate it in terms of the enstrophy $||\nabla|^\alpha u|_2^2$. We have

$$\begin{aligned}
|B_1| &\lesssim \sum_{\substack{p',p''\geq q_j\\ |p'-p''|\leq 2}} |u_{p'} \otimes u_{p''} : \nabla U_{q_j}| + \sum_{\substack{p',p''\geq q_j\\ |p'-p''|\leq 2}} |u_{p'} \otimes U_{p''} : \nabla U_{q_j}| \\
&\leq \lambda_{q_j}^{2\alpha} |u_{\geq q_j}|_2^2 + \lambda_{q_j}^{2\alpha} |u_{\geq q_j}|_2 |U_{\geq q_j}|_2 \\
&\leq ||\nabla|^\alpha u_{\geq q_j}|_2^2 + \lambda_{q_j}^{3\alpha-5/2} ||\nabla|^\alpha u_{\geq q_j}|_2 \\
&\leq ||\nabla|^\alpha u_{\geq q_j}|_2^2 + \lambda_{q_j}^{6\alpha-5-1/2} ||\nabla|^\alpha u_{\geq q_j}|_2.
\end{aligned}$$

Again, using Lemma 2.1, Bernstein and Hölder inequalities we obtain

$$\begin{aligned}
|B_2| &= \left|U_{q_j} \otimes \tilde{w}_{q_j} : \nabla u_{\leq q_j}\right| \leq |U_{q_j}|_2 |\tilde{w}_{q_j}|_\infty |\nabla u_{\leq q_j}|_2 \\
&\leq \lambda_{q_j}^{2\alpha-5/2} |\tilde{w}_{q_j}|_\infty ||\nabla|^\alpha u|_2 \leq \lambda_{q_j}^{4\alpha-7/2} ||\nabla|^\alpha u|_2 \leq \lambda_{q_j}^{6\alpha-5-1/2} ||\nabla|^\alpha u|_2. \\
|B_3| &\leq |\tilde{u}_{q_j}|_2 |w_{\leq q_j}|_\infty |\nabla U_{q_j}|_2 \lesssim \lambda_{q_j}^{2\alpha-3/2} |\tilde{u}_{q_j}|_2 \sum_{p\leq q_j} |w_p|_\infty \\
&\lesssim \lambda_{q_j}^{3\alpha-5/2} ||\nabla|^\alpha u|_2 \leq \lambda_{q_j}^{6\alpha-5-1/2} ||\nabla|^\alpha u|_2.
\end{aligned}$$

We thus obtain

$$|B| \lesssim ||\nabla|^\alpha u_{\geq q_j}|_2^2 + \lambda_{q_j}^{6\alpha-5-1/2} ||\nabla|^\alpha u|_2. \tag{11}$$

Continuing in a similar fashion we write

$$\begin{aligned}
C = \sum_{\substack{p',p''\geq q_j\\ |p'-p''|\leq 2}} & u_{p'} \otimes u_{p''} : \nabla w_{q_j} + u_{\leq q_j} \otimes \tilde{u}_{q_j} : \nabla w_{q_j} \\
&+ \tilde{u}_{q_j} \otimes u_{\leq q_j} : \nabla w_{q_j} - \text{repeated} = C_1 + C_2 + C_3.
\end{aligned}$$

We have

$$|C_1| \leq |\nabla w_{q_j}|_\infty |u_{\geq q_j}|_2^2 \lesssim \delta ||\nabla|^\alpha u|_2^2.$$

In C_2 we move the derivative onto $u_{\leq q_j}$ and estimate as usual,

$$|C_2| \leq |\nabla u|_2 |\tilde{u}_{q_j}|_2 |w_{q_j}|_\infty \lesssim ||\nabla|^\alpha u|_2 |\tilde{u}_{q_j}|_2 \lambda_{q_j}^{2\alpha-1} \leq ||\nabla|^\alpha u|_2^2 \lambda_{q_j}^{6\alpha-5-1/2}.$$

Using a uniform bound on the energy we have for C_3,

$$|C_3| \lesssim \lambda_{q_j} |w_{q_j}|_\infty |\tilde{u}_{q_j}|_2 \leq \delta \lambda_{q_j}^\alpha ||\nabla|^\alpha \tilde{u}_{q_j}|_2 \leq \delta \lambda_{q_j}^{6\alpha-5} ||\nabla|^\alpha \tilde{u}_{q_j}|_2.$$

Thus,

$$|C| \lesssim \delta ||\nabla|^\alpha u|_2^2 + ||\nabla|^\alpha u|_2^2 \lambda_{q_j}^{6\alpha-5-1/2} + \delta \lambda_{q_j}^{6\alpha-5} ||\nabla|^\alpha \tilde{u}_{q_j}|_2. \tag{12}$$

Now combining estimates (10), (11), (12) along with the boundedness of $E(t_0)$ we obtain

(13)
$$\int_0^{t_0} |A+B+C|\,ds \lesssim \delta\lambda_{q_j}^{6\alpha-5}t_0 + \int_0^{t_0} ||\nabla|^\alpha u_{\geq q_j}|_2^2\,ds + E(t_0)^{1/2}t_0^{1/2}\lambda_{q_j}^{6\alpha-5-1/2} + \delta E(t_0) + \delta\lambda_{q_j}^{6\alpha-5}\int_0^{t_0} ||\nabla|^\alpha \tilde{u}_{q_j}|_2\,ds.$$

And for large j, and fixed t_0, this gives

$$\int_0^{t_0} |A+B+C|\,ds \lesssim \delta\lambda_{q_j}^{6\alpha-5}t_0 + \frac{\nu}{2}E(t_0).$$

Pugging this back into (9) gives the estimate

$$|\tilde{u}_{q_j}(t_0)|_2^2 \gtrsim \lambda_{q_j}^{6\alpha-5},$$

for all $j > j_0$, which shows that $u(t_0)$ has infinite energy, a contradiction.

The last statement of the proposition follows from the fact that we have the bounds on $|u(t)|_2 \leq |U|_2$ and $E(t_0) \leq (2\nu)^{-1}|U|_2^2$ which remove dependence of the constants on u. $\square$

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