ILL-POSEDNESS FOR SUBCRITICAL HYPERDISSIPATIVE NAVIER-STOKES EQUATIONS IN THE LARGEST CRITICAL SPACES

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ABSTRACT. We study the incompressible Navier-Stokes equations with a fractional Laplacian and prove the existence of discontinuous Leray-Hopf solutions in the largest critical space with arbitrarily small initial data.

1. INTRODUCTION

In this paper we study the supercritical 3D Navier-Stokes equations with a fractional power of the Laplacian

\[
\begin{aligned}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p &= -\nu (-\Delta)^{\alpha} u, \quad x \in \mathbb{T}^3, \quad t \geq 0, \\
\nabla \cdot u &= 0, \\
u(0) &= u_0,
\end{aligned}
\]

(1)

where the velocity \( u(x, t) \) and the pressure \( p(x, t) \) are unknowns, \( u_0 \in L^2(\mathbb{T}^3) \) is the initial condition, \( \nu > 0 \) is the kinematic viscosity coefficient of the fluid, and \( \alpha > 0 \). The case \( \alpha = 1 \) corresponds to the classical Navier-Stokes equations, which has been studied extensively for decades. We refer to [7, 17] for the classical theory for these equations. In the case \( \alpha \geq 5/4 \) the equations are well-posed, as the dissipative term simply dominates the nonlinear term. Moreover, the global regularity is known even in a slightly supercritical case, i.e., when logarithmic corrections to the Fourier multiplier of the dissipative term are present (see [16, 4]). However, a finite time blow up of solutions to (1) remains a possibility for \( \alpha < 5/4 \) due to a supercritical nature of the equations. Nevertheless, a partial regularity result [3] has been established in the supercritical case \( \alpha = 1 \), later extended to \( \alpha \in (1, 5/4) \) in [11]. There are also various regularity criteria in the case \( \alpha = 1 \), most of which are of Ladyzhenskaya-Prodi-Serrin type [8, 13, 14, 15, 10, 6, 4], which can also be extended to \( \alpha \in (1, 5/4) \).

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One of the open questions studied extensively is whether solutions bounded in the largest critical case ($\dot{B}^{-1}_{\infty,\infty}$ for $\alpha = 1$) are regular. A positive answer to this question would extend the famous $L_t^\infty L_x^3$ result due to I. I. I. Seregin, and Šverák [10]. In addition, the best small initial result for the 3D NSE, due to Koch and Tataru [12], is in the space $BMO^{-1}$, and it is not known either if its extension to the $B^{-1}_{\infty,\infty}$ is possible.

In view of these problems two “negative” results have been obtained in the space $\dot{B}^{-1}_{\infty,\infty}$. First, Bourgain and Pavlovic [2] proved that there are solutions to the 3D NSE equations, with arbitrary small initial data in $\dot{B}^{-1}_{\infty,\infty}$ that become arbitrarily large in $\dot{B}^{-1}_{\infty,\infty}$ in arbitrarily small time. Second, Leray-Hopf solutions with arbitrary small initial data, but discontinuous in $B^{-1}_{\infty,\infty}$ were obtained in [5].

The largest critical space for the fractional NSE (1) is $\dot{B}^{1-2\alpha}_{\infty,\infty}$. Recently Yu and Zhai [18] obtained a small initial data result in this space in the hypodissipative case $\alpha \in (1/2, 1)$. Heuristically, the hypodissipative NSE behaves better because it is closer to the fractional heat semigroup in critical spaces. In the hyperdissipative case it is therefore natural to expect ill-posedness results of the type mentioned above. Indeed, in this paper we demonstrate this in the case $\alpha \in [1, 5/4)$ by constructing a Leray-Hopf solution with arbitrarily small initial data, which is discontinuous in the critical Besov space $B^{1-2\alpha}_{\infty,\infty}$. It is thus a direct extension of our previous result stated in [5]. The method breaks down either when $\alpha$ passes beyond the value of 1, which is consistent with the result of Yu and Zhai, and at $5/4$ and beyond, which is consistent with the global regularity in that range.

We now fix our notation. We assume periodic boundary conditions in all 3 dimensions, so $\mathbb{T}^3$ will denote the 3D torus, while $|\cdot|_p$, $p \geq 1$, denotes the $L^p$-norm in $\mathbb{T}^3$. We let $\hat{f}$ and $\tilde{f}$ stand for the forward and, respectively, inverse Fourier transforms on the torus. The Fourier multiplier with symbol $|\xi|^{\alpha}$, where $\xi$ stands for the frequency vector and $\alpha > 0$, is denoted by $|\nabla|^{\alpha}$. The fractional Laplacian operator $(-\Delta)^\alpha$ has symbol $|\xi|^{2\alpha}$. We write $p(\xi) = \text{id} - |\xi|^{-2}\xi \otimes \xi$, $\xi \neq 0$, $p(0) = \text{id}$, for the symbol of the Leray-Hopf projection on the divergence-free fields. We fix notation for the dyadic a-dimensional wavenumbers $\lambda_q = 2^q$. We use extensively the classical dyadic decomposition throughout: $u = \sum_{q \geq 0} u_q$, where $u_q$ is the Littlewood-Paley projection with the Fourier support contained in $\{\lambda_{q-1} < |\xi| < \lambda_{q+1}\}$. The definitions are standard and can be found in the references above. We often will be using the extended projection defined by $\tilde{u}_q = u_{q-1} + u_q + u_{q+1}$, $q \geq 1$, and projection onto the dyadic ball, $u_{\leq q} = \sum_{p=0}^q u_p$. Thus, $u_q$ is
supported on \( \{ \lambda_{q-2} < |\xi| < \lambda_{q+2} \} \) and we have the identity
\[
(2) \quad \int_{\mathbb{T}^3} u \cdot u_q \, dx = \int_{\mathbb{T}^3} \tilde{u}_q \cdot u_q \, dx.
\]
With the Littlewood-Paley decomposition we define Besov spaces \( B^s_{r,\infty} \), \( s \in \mathbb{R} \), \( r \geq 1 \) by requiring
\[
\|u\|_{B^s_{r,\infty}} = \sup_{q \geq 0} \lambda^s_q \|u_q\|_r < \infty.
\]
We will frequently refer to Bernstein’s inequalities, which state that for all \( 1 \leq r < r' \leq \infty \), and in three dimensions, one has
\[
|u_q|_{r'} \lesssim \lambda^3 \left( \frac{1}{r} - \frac{1}{r'} \right) q |u_q|_r,
\]
where here and throughout \( \lesssim \) denote inequality up to an absolute constant. Finally, let \( \vec{e}_1, \vec{e}_2, \) etc., stand for the vectors of the standard unit basis.

2. ILL-POSEDNESS OF NSE

The Navier-Stokes equation with a fractional power of the Laplacian is given by
\[
(3) \quad u_t + (u \cdot \nabla)u = -\nu (-\Delta)^\alpha u - \nabla p.
\]
Here \( u \) is a three dimensional divergence free field on \( \mathbb{T}^3 \), and \( \alpha \in [1, 5/4) \). Let us recall that for every field \( U \in L^2(\mathbb{T}^3) \) there exists a weak solution \( u \in C_w([0,T); L^2) \cap L^2([0,T); H^1) \) to (3) such that the energy inequality
\[
(4) \quad \|u(t)\|^2_2 + 2\nu \int_0^t \|\nabla^\alpha u(s)\|^2 ds \leq \|U\|^2_2,
\]
holds for all \( t > 0 \) and \( u(t) \to U \) strongly in \( L^2 \) as \( t \to 0 \). In what follows we do not actually use inequality (4) which allows us to formulate a more general statement below in Proposition 2.2.

Let us choose a strictly increasing sequence \( \{ q_j \} \in \mathbb{N} \) with elements sufficiently far apart so that at least \( \lambda_{q_{j+1}}^2 \lambda_{q_{j+1}}^-5 < 1 \). We consider the following lattice blocks:
\[
A_j = \left[ \frac{9}{10} \lambda_{q_j}, \frac{11}{10} \lambda_{q_j} \right] \times \left[ -\frac{1}{10} \lambda_{q_j}, \frac{1}{10} \lambda_{q_j} \right]^2 \cap \mathbb{Z}^3
\]
\[
B_j = \left[ -\frac{1}{10} \lambda_{q_j-1}, \frac{1}{10} \lambda_{q_j-1} \right]^2 \times \left[ \frac{9}{10} \lambda_{q_j-1}, \frac{11}{10} \lambda_{q_j-1} \right] \cap \mathbb{Z}^3
\]
\[
C_j = A_j + B_j
\]
\[
A_j^* = -A_j, B_j^* = -B_j, C_j^* = -C_j.
\]
Thus, \( A_j, C_j \) and their conjugates lie in the \( q_j \)-th shell, while \( B_j, B_j^* \) lie in the adjacent \( (q_j - 1) \)-th shell. The particular choice of scaling exponents
9/10, 11/10, etc., is unimportant as long as the blocks fit into the their respective shells. Let us denote
\[ \vec{e}_1(\xi) = p(\xi)\vec{e}_1, \quad \vec{e}_2(\xi) = p(\xi)\vec{e}_2. \]
We now define the initial condition field to be the following sum
\[ U = \sum_{j \geq 1} (U_{q_j} + U_{q_j-1}), \]
where the components, on the Fourier side, are
\[ \hat{U}_{q_j}(\xi) = \lambda_{q_j}^{2\alpha-4} \left( \vec{e}_2(\xi)\chi_{A_j \cup A_j^*} + i(\vec{e}_2(\xi) - \vec{e}_1(\xi))\chi_{C_j} - i(\vec{e}_2(\xi) - \vec{e}_1(\xi))\chi_{C_j^*} \right), \]
and
\[ \hat{U}_{q_j-1}(\xi) = \lambda_{q_j}^{2\alpha-4}\vec{e}_1(\xi)\chi_{B_j \cup B_j^*}. \]
By construction, \( \hat{U}(\xi) = \hat{U}(\xi) \), which ensures that \( U \) is real. Since \( U \) has no modes in the \((q_j + 1)\)-st shell, then the extended Littlewood-Paley projection of the \( j \)-th component has the form \( \hat{U}_{q_j} = U_{q_j-1} + U_{q_j} \).

Lemma 2.1. We have \( U \in B^{1+\frac{3}{r}-2\alpha}_{r,\infty} \), for any \( 1 < r \leq \infty \).

Proof. We give the estimate only for one block, the other ones being similar. Using boundedness of the Leray-Hopf projection, we have, for all \( 1 < r < \infty \),
\[ |\lambda_{q_j}^{2\alpha-4}(\vec{e}_2(\cdot)\chi_{A_j})^\vee|_r \lesssim \lambda_{q_j}^{2\alpha-4}|(\chi_{A_j})^\vee|_r. \]
Notice that by construction,
\[ |(\chi_{A_j})^\vee(x_1, x_2, x_3)| = |D_{(c+1)\lambda_{q_j}}(x_1)D_{c\lambda_{q_j}}(x_2)D_{c\lambda_{q_j}}(x_3)|, \]
where \( D_N \) denotes the Dirichlet kernel. Hence,
\[ |(\chi_{A_j})^\vee|_r \leq |D_{(c+1)\lambda_{q_j}}|_r|D_{c\lambda_{q_j}}|_r^2. \]
By a well-known estimate, we have \( |D_N|_r \leq N^{1-\frac{1}{r}} \) (c.f. [9]). Putting the above estimates together implies the desired inclusion in \( B^{1+3/r-2\alpha}_{r,\infty} \). In the case \( r = \infty \) we simply use the triangle inequality to obtain
\[ |U_{q_j}|_\infty \lesssim \lambda_{q_j}^{2\alpha-1}. \]

Let us now examine the trilinear term. We will use the following notation for convenience
\[ u \otimes v : \nabla w = \int_{T^3} v_i \partial_i w_j u_j dx. \]
Using the antisymmetry we obtain
\[
U \otimes U : \nabla U_{q_j} = \sum_{k \geq j+1} \bar{U}_{q_k} \otimes \bar{U}_{q_k} : \nabla U_{q_j} + \bar{U}_{q_j} \otimes \bar{U}_{q_j} : \nabla U_{q_j}
+ U_{\leq q_{j-1}} \otimes \bar{U}_{q_j} : \nabla U_{q_j} + \bar{U}_{q_j} \otimes U_{\leq q_{j-1}} : \nabla U_{q_j}
= \sum_{k \geq j+1} \bar{U}_{q_k} \otimes \bar{U}_{q_k} : \nabla U_{q_j} + U_{q_{j-1}} \otimes U_{q_j} : \nabla U_{q_j}
- U_{q_j} \otimes U_{q_j} : \nabla U_{\leq q_{j-1}}
= A + B + C.
\]

Using Bernstein’s inequalities we estimate
\[
|A| \lesssim \lambda_{q_j} |U_{q_j}|_\infty \sum_{k \geq j+1} |\bar{U}_{q_k}|^2 \lesssim \lambda_{q_j}^{2\alpha} \lambda_{q_{j+1}}^{4\alpha-5} \leq 1,
\]
\[
|C| \lesssim |U_{q_j}|^2 \sum_{k \leq j-1} \lambda_{q_k} |\bar{U}_{q_k}|_\infty \lesssim \lambda_{q_{j-1}}^{2\alpha} \lambda_{q_j}^{4\alpha-5} \leq 1,
\]
where in the latter inequality we used the fact $|U_{q_j}|^2 \sim \lambda_{q_j}^{2\alpha-5/2}$. On the other hand, a straightforward computation shows that
\[
B \sim \lambda_{q_j}^{6\alpha-5},
\]
which is thus the dominant term of the three, and hence,
\[
U \otimes U : \nabla U_{q_j} \sim \lambda_{q_j}^{6\alpha-5}.
\]

**Proposition 2.2.** Let $u \in C_w([0, T); L^2) \cap L^2([0, T); H^1)$ be a weak solution to the NSE with initial condition $u(0) = U$. Then there is $\delta = \delta(u) > 0$ such that
\[
\limsup_{t \to 0^+} \|u(t) - U\|_{B^{1-2\alpha}_{\infty, \infty}} \geq \delta.
\]

If, in addition, $u$ is a Leray-Hopf solution satisfying the energy inequality (4), then $\delta$ can be chosen independent of $u$.

**Proof.** Let us test (3) with $u_{q_j}$. Using (2), we find
\[
\partial_t (\bar{u}_{q_j} \cdot u_{q_j}) = -\nu |\nabla|^{\alpha} \bar{u}_{q_j} \cdot |\nabla|^{\alpha} u_{q_j} + u \otimes u : \nabla u_{q_j},
\]
where as defined before, $\bar{u}_{q_j} = u_{q_j-1} + u_{q_j} + u_{q_j+1}$. Denoting $E(t) = \int_0^t \|\nabla|^{\alpha} u\|^2_{L^2} ds$ we obtain
\[
|\bar{u}_{q_j}(t)|^2 \geq |U_{q_j}|^2 - \nu E(t) + c_1 \lambda_{q_j}^{6\alpha-5} t
- c_2 \int_0^t \|u \otimes u : \nabla u_{q_j} - U \otimes U : \nabla U_{q_j}\|_1 ds,
\]
for some positive constants \(c_1\) and \(c_2\). We now show that if the conclusion of the proposition fails then for some small \(t > 0\) the integral term the growth of the integral term above becomes less than \(c_1\lambda^{6\alpha-5}t\) for large \(j\). This forces \(\|\tilde{u}_{q_j}(t)\|_2^2 \gtrsim \lambda^{6\alpha-5}t\) for all large \(j\). Hence \(u\) has infinite energy, which is a contradiction.

So suppose that for every \(\delta > 0\) there exists \(t_0 = t_0(\delta) > 0\) such that \(\|u(t) - U\|_{B^{1-2\alpha}_{\infty}\infty} < \delta\) for all \(0 < t \leq t_0\). Denoting \(w = u - U\) we write

\[
  u \otimes u : \nabla u_{q_j} - U \otimes U : \nabla U_{q_j} = w \otimes U : \nabla U_{q_j} + u \otimes w : \nabla U_{q_j}
  + u \otimes u : \nabla w_{q_j} = A + B + C.
\]

We will now decompose each triplet into three terms according to the type of interaction (c.f. Bony [1]) and estimate each of them separately.

\[
  A = \sum_{p', p'' \geq q_j \atop |p' - p''| \leq 2} w_{p'} \otimes U_{p''} : \nabla U_{q_j} + w_{\leq q_j} \otimes \tilde{U}_{q_j} : \nabla U_{q_j}
  + \tilde{w}_{q_j} \otimes U_{\leq q_j} : \nabla U_{q_j} - \text{repeated} = A_1 + A_2 + A_3.
\]

Let us fix \(r \in (1, 3/(4\alpha - 2))\) and use Lemma 2.1 along with Hölder and Bernstein’s inequalities to estimate \(A_1\):

\[
  |A_1| \leq |\nabla U_{q_j}|_{r'} \sum |w_{p'}|_\infty |U_{p''}|_r \lesssim \lambda_{q_j}^{2\alpha-3+\frac{3}{r}} \sum |w_{p'}|_\infty \lambda_{p''}^{2\alpha-1-\frac{2}{r}} \lesssim \delta \lambda_{q_j}^{2\alpha-3+\frac{3}{r}} \leq \delta \lambda_{q_j}^{6\alpha-5}.
\]

Intergrating by parts we obtain \(A_2 = U_{q_j} \otimes \tilde{U}_{q_j} : \nabla w_{\leq q_j}\). Thus, using the same tools,

\[
  |A_2| \leq |\tilde{U}_{q_j}|_2^2 |\nabla w_{\leq q_j}|_{\infty} \lesssim \lambda_{q_j}^{4\alpha-5} \sum_{p \leq q_j} \lambda_p |w_p|_{\infty} < \delta \lambda_{q_j}^{6\alpha-5}.
\]

And finally,

\[
  |A_3| \leq \lambda_{q_j} |U_{\leq q_j}|_2 |U_{q_j}|_2 |\tilde{w}_{q_j}|_{\infty} \lesssim \lambda_{q_j}^{4\alpha-4} |\tilde{w}_{q_j}|_{\infty} < \delta \lambda_{q_j}^{6\alpha-5}.
\]

We have shown the following estimate:

\[(10)\]

\[
  |A| \lesssim \delta \lambda_{q_j}^{6\alpha-5}.
\]

As to \(B\) we decompose analogously,

\[
  B = \sum_{p', p'' \geq q_j \atop |p' - p''| \leq 2} u_{p'} \otimes w_{p''} : \nabla U_{q_j} + u_{\leq q_j} \otimes \tilde{w}_{q_j} : \nabla U_{q_j}
  + \tilde{u}_{q_j} \otimes w_{\leq q_j} : \nabla U_{q_j} - \text{repeated} = B_1 + B_2 + B_3.
\]
The term $B$ is the least problematic. Here we do not even have to use the smallness of $w$ and can just roughly estimate it in terms of the enstrophy $\|\nabla^\alpha u\|_2^2$. We have

$$
|B_1| \lesssim \sum_{j : p, p' \geq j, |p'| \leq 2} |u_{p'} \otimes u_{p''} : \nabla w_j| + \sum_{j : p, p' \geq j, |p'| \leq 2} |u_{p'} \otimes U_{p''} : \nabla U_{j_2}| \\
\leq \lambda^{2\alpha}_j \|u_{\geq j_2}\|_2^2 + \lambda^{2\alpha}_j \|u_{\geq j_2}\|_2 \|U_{\geq j_2}\|_2 \\
\leq \|\nabla^\alpha u_{\geq j_2}\|_2^2 + \lambda^{3\alpha-5/2}_j \|\nabla^\alpha u_{\geq j_2}\|_2 \\
\leq \|\nabla^\alpha u_{\geq j_2}\|_2^2 + \lambda^{6\alpha-5-1/2}_j \|\nabla^\alpha u_{\geq j_2}\|_2.
$$

Again, using Lemma 2.1, Bernstein and Hölder inequalities we obtain

$$
|B_2| = |U_{q_2} \otimes \tilde{w}_{q_2} : \nabla u_{\leq q_2}| \leq |U_{q_2}|_2 \|\tilde{w}_{q_2}\|_\infty \|\nabla u_{\leq q_2}\|_2 \\
\leq \lambda^{2\alpha-5/2}_j \|\tilde{w}_{q_2}\|_\infty \|\nabla^\alpha u\|_2 \leq \lambda^{2\alpha-7/2}_j \|\nabla^\alpha u\|_2 \leq \lambda^{6\alpha-5-1/2}_j \|\nabla^\alpha u\|_2.
$$

Thus we obtain

(11) $|B| \lesssim \|\nabla^\alpha u_{\geq j_2}\|_2^2 + \lambda^{6\alpha-5-1/2}_j \|\nabla^\alpha u\|_2.$

Continuing in a similar fashion we write

$$
C = \sum_{j : p, p' \geq j, |p'| \leq 2} u_{p'} \otimes u_{p''} : \nabla w_{q_j} + u_{\leq q_j} \otimes \tilde{u}_{q_j} : \nabla w_{q_j} \\
+ \tilde{u}_{q_j} \otimes u_{\leq q_j} : \nabla w_{q_j} - \text{repeated} = C_1 + C_2 + C_3.
$$

We have

$$
|C_1| \leq \|\nabla w_{q_j}\|_\infty \|u_{\geq j_2}\|_2^2 \lesssim \delta \|\nabla^\alpha u\|_2^2.
$$

In $C_2$ we move the derivative onto $u_{\leq q_j}$ and estimate as usual,

$$
|C_2| \leq \|\nabla u\|_2 \|\tilde{u}_{q_j}\|_2 \|w_{q_j}\|_\infty \lesssim \|\nabla^\alpha u\|_2 \|\tilde{u}_{q_j}\|_2 \lambda^{2\alpha-1}_j \leq \|\nabla^\alpha u\|_2^2 \lambda^{6\alpha-5-1/2}_j.
$$

Using a uniform bound on the enstrophy we have for $C_3$,

$$
|C_3| \lesssim \lambda_j \|u_{q_j}\|_\infty \|\tilde{u}_{q_j}\|_2 \leq \delta \lambda^{\alpha}_j \|\nabla^\alpha \tilde{u}_{q_j}\|_2 \leq \delta \lambda^{6\alpha-5}_j \|\nabla^\alpha \tilde{u}_{q_j}\|_2.
$$

Thus,

(12) $|C| \lesssim \delta \|\nabla^\alpha u\|_2^2 + \|\nabla^\alpha u\|_2^2 \lambda^{6\alpha-5-1/2}_j + \delta \lambda^{6\alpha-5}_j \|\nabla^\alpha \tilde{u}_{q_j}\|_2.$
Now combining estimates (10), (11), (12) along with the boundedness of $E(t_0)$ we obtain

\begin{equation}
\int_0^{t_0} |A + B + C| \, ds \lesssim \delta \lambda_{q_j}^{6\alpha - 5} t_0 + \int_0^{t_0} \left\| \nabla |^\alpha u_{\geq q_j} \right\|_2^2 \, ds + E(t_0)^{1/2} t_0^{1/2} \lambda_{q_j}^{6\alpha - 5 - 1/2} \\
+ \delta E(t_0) + \delta \lambda_{q_j}^{6\alpha - 5} \int_0^{t_0} \left\| \nabla |^\alpha \tilde{u}_{q_j} \right\|_2 \, ds.
\end{equation}

And for large $j$, and fixed $t_0$, this gives

\[
\int_0^{t_0} |A + B + C| \, ds \lesssim \delta \lambda_{q_j}^{6\alpha - 5} t_0 + \frac{\nu}{2} E(t_0).
\]

Pugging this back into (9) gives the estimate

\[
|\tilde{u}_{q_j}(t_0)|_2^2 \gtrsim \lambda_{q_j}^{6\alpha - 5},
\]

for all $j > j_0$, which shows that $u(t_0)$ has infinite energy, a contradiction.

The last statement of the proposition follows from the fact that we have the bounds on $|u(t)|_2 \leq |U|_2$ and $E(t_0) \leq (2\nu)^{-1} |U|_2^2$ which remove dependence of the constants on $u$. \hfill \Box

\textbf{References}


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