"Sometimes the answer is not a
theorem, and a theorem is not
the answer."

Charlie Doering

VOLUMETRIC THEORY OF INTERMITTENCY IN FULLY DEVELOPED TURBULENCE

ALEXEY CHESKIDOV AND ROMAN SHVYDKOY

ABSTRACT. This study introduces a new family of volumetric flatness factors which give a rigorous parametric description of the phenomenon of intermittency in fully developed turbulent flows. These quantities gather information about the most "active" part of a velocity field at each scale ℓ, and allows one to define a dimension function $p \to D_p$ that recovers intermittency correction to the structure exponents ζ_p in an explicit way. In particular, the predictions of the Frisch-Parisi multifractal formalism can be recovered in a systematic and rigorous way.

Within this framework we identify active regions that carry the most energetic part of a velocity field at a given scale ℓ. A threshold for what constitutes to be active is defined explicitly. Active regions have proven to be experimentally observable in our previous joint work [NLK$^+$19], and shown to capture concentration of the energy cascade as $\ell \to 0$, in [CS14a].

We present several examples of fields which exhibit arbitrary multifractal spectrum within theoretically permitted limitations. At the same time we demonstrate with the use of a probabilistic argument that a random field is expected to produce the classical K41 spectrum in the limit $\ell \to 0$. Intermittent deviations from K41 theory are estimated at any finite scale also.

Lastly, we present a detailed information-theoretic analysis of the introduced objects. In particular, we quantify concentration of a given source-field in terms of the volume factors, thresholds, and active regions.

CONTENTS

1. INTRODUCTION

The notion of a structure function goes back to the seminal work of Kolmogorov [Kol91]. For a length scale ℓ, the nth order structure function is defined is

$$(1) \qquad S_p(\ell) = \langle |\delta_\ell \mathbf{u}|^p \rangle = \langle |\mathbf{u}(r + \vec{\ell}, t) - \mathbf{u}(r, t)|^p \rangle,$$

with some appropriate average $\langle \cdot \rangle$ where $|\vec{\ell}| = \ell$. It is customary to represent the structure functions by the scaling law

$$(2) \qquad S_p(\ell) \sim (\varepsilon \ell)^{\frac{p}{3}} \left(\frac{\ell}{L} \right)^{\zeta_p - \frac{p}{3}} \sim U^p \left(\frac{\ell}{L} \right)^{\zeta_p},$$

where U and L are characteristic velocity and length scale, ζ_p are scaling exponents, and ε is the average energy dissipation rate predicted to be independent of the Reynolds number by Kolmogorov's theory of turbulence. More precisely,

$$\frac{\varepsilon L}{U^3} \sim Re^0,$$

for large Re, which has been proved in one direction (as an upper bound) for long time averages of solutions to the forced 3D Navier-Stokes equations [DF02], while the other direction remains a major open problem in mathematical theory of turbulence.

Kolmogorov's theory of turbulence also predicts self-similar scaling $\delta_\ell \mathbf{u} \sim \ell^{1/3}$ and, as a consequence, the law

$$(3) \qquad \zeta_p = \frac{p}{3}.$$

In fact, for the longitudinal third order structure function $S_3^{\|}(\ell) = \langle (\delta_\ell \mathbf{u} \cdot \ell / |\ell|)^3 \rangle$ this is known as the Kolmogorov's $\frac{4}{5}$th law and can be derived from the first principles: $S_3^{\|}(\ell) = -\frac{4}{5} \varepsilon \ell$, see [Fri95] and Section 2.5 below. The validity of (3) has been challenged almost immediately by Landau due to intermittency – non-uniform distribution of energy transfer from one scale to another – although the first credible experimental confirmation of this phenomenon came much later in the work of Anselmet et. al. [AGHA84]. We refer to [ES06, Fri95] for detailed historic surveys of the subject. While deviations from the law (3) have now being supported by numerous data, see [SS93, SVB+96, SKF+16, NLK+19, ISY20] for the latest, the exact formula for ζ_p remains unknown.

Several phenomenological models have been invented to account for intermittency. At the core of these models lies the notion of a *spatial intermittency dimension D* at scale ℓ, a number between 0 and 3, such that

$$\text{Number of active eddies} \sim \left(\frac{L}{\ell} \right)^D.$$

Here L is the size of the domain, and by "active eddies" we understand fluid blobs of size ℓ of the filtered field that participate in the energy transfer to the next scale $\ell/2$. The case $D = 3$ corresponds to Kolmogorov's regime where the whole region is actively involved in the transfer at each scale in the inertial range. The other borderline case is $D = 0$ (extreme intermittency), where the number of eddies involved is bounded over all the scales. Intermittent flows, where $D < 3$, exhibit deviations from Kolmogorov's regime by about 10% experimentally, see the literature above. If defined in rigorous mathematical terms, the notion of intermittency can be useful in formulating effective regularity criteria for solutions to the classical incompressible fluid models, such as the Navier-Stokes equation. In [CS14a] the definition was proposed based on the Littlewood-Paley decomposition. More precisely, localizing the fluid velocity $\mathbf{u}$ in frequency by taking the Littlewood-Paley projection $\mathbf{u}_\lambda$ to the dyadic shell of radius $\lambda = \ell^{-1}$, eddies can be defined as active regions where $\mathbf{u}_\lambda$ is large is some L^p norm. Then counting the number of eddies at each scale leads to the dimension as a saturation parameter $D_{q,p}$ in the classical Bernstein inequality

$$(4) \qquad \|\mathbf{u}_\lambda\|_{L^q} = \lambda^{(3 - D_{q,p})\left(\frac{1}{p} - \frac{1}{q} \right)} \|\mathbf{u}_\lambda\|_{L^p}.$$

This notion of intermittency when applied to Leray-Hopf solutions of the Navier-Stokes equations, with appropriate temporal averaging, leads to numerous rigorous results [CS14a, CS14b, CD19, CD21]. For

instance, $D_{\infty,2} \geqslant 3/2$ implies regularity of solutions, [CS14b]. In a parallel series of works, a similar in spirit theory based on the concept of sparseness was developed by Grujic et. al. in [BadVGG18, BFG19].

While (4) is suitable for mathematical analysis, in physics a well-accepted measure of intermittency is the multifractal spectrum (MFR) introduced by Frisch an Parisi [PF83] – the dimension $d_h(\ell)$ of a set where the Hölder exponent h is attained by a fluid flow at scale ℓ. In this paper we introduce a new analytic framework based on a suitably defined volumetric quantities associated with the field $\delta_\ell \mathbf{u}$ that reconciles the two approaches in a rigorous way. The main idea is based on viewing velocity increments $\delta_\ell \mathbf{u}$ as a source of information whose concentration in active regions A_p is measured by explicitly defined dimensions D_p and then translated into intermittency corrections to the structure functions S_p. Moreover, we show that D_p coincides with the MFR spectrum d_h under the transformation $h = \zeta_p'$.

To describe the main ingredients of the framework more specifically let us pass to the adimensional variables $\mathbf{u} \to \mathbf{u}/U$ and $\ell \to \ell/L$ and consider the scailing exponents defined according to (2)

$$\zeta_p := \log_\ell \langle |\delta_\ell \mathbf{u}|^p \rangle,$$

where $\langle |\delta_\ell \mathbf{u}|^p \rangle$ is the appropriate average (e.g., in space and angles) of the velocity increments at length scale ℓ, even though the precise nature of the localisation of $\mathbf{u}$ is not important for many of our results. As a function of p, the scaling exponent ζ_p is smooth, concave, with $\zeta_0 = 0$. Then for Lebesgue exponents $p, q \in \mathbb{R}$ we define the (q,p)-*volume factor* and dimension $D_{q,p}$ at scale ℓ,

$$(5) \qquad V_{q,p}(\ell) := \frac{\langle |\delta_\ell \mathbf{u}|^q \rangle^{\frac{p}{p-q}}}{\langle |\delta_\ell \mathbf{u}|^p \rangle^{\frac{q}{p-q}}}, \qquad D_{q,p}(\ell) := 3 - \log_\ell V_{q,p},$$

consistent with (4). Remarkably, in the limit, the intermittency dimension $D_p := \lim_{q \to p} D_{q,p}$ turns out to be

$$D_p = 3 - \zeta_p + p\zeta_p',$$

and hence the dimension D_p coincides with the MFR dimension d_h defined via Legendre transform

$$(6) \qquad D_{p(h)} = d_h := \inf_p (3 + hp - \zeta_p).$$

Indeed, $D_p = d_h$ at the point where the infimum is attained, i.e., $h = \zeta_p' = \frac{\langle |\delta_\ell \mathbf{u}|^p \log_\ell |\delta_\ell \mathbf{u}| \rangle}{\langle |\delta_\ell \mathbf{u}|^p \rangle}$. The resulting map $p \to D_p$ reproduces the same multifractal spectrum as $h \to d_h$ but now in terms of the same Lebesgue variable p that parameterises the structure exponents ζ_p. This allows us to derive an explicit intermittency correction to the classical linear profile (3):

$$\zeta_p = p\zeta_0' + \mathcal{I}_p, \quad \text{where} \quad \mathcal{I}_p = p \int_0^p \frac{D_s - 3}{s^2} \, \mathrm{d}s.$$

In Subsection 2.2 we describe functional properties of ζ_p, D_p, and d_h. In particular, all possible scenarios are classified at two endpoints $(p_{\max}, h_{\min})$ and $(p_{\min}, h_{\max})$.

The dimension D_p is naturally connected to the active volume

$$V_p := \lim_{q \to p} V_{q,p} = \ell^{3 - D_p},$$

which is an upper bound on the size of the active region A that can be parameterized in terms of the Hölder or Lebesgue exponents as

$$A_h^{\text{Hölder}} := \{x : c\ell^h \leqslant |\delta_\ell \mathbf{u}(x)| \leqslant C\ell^h\} = A_p^{\text{Lebesgue}} := \{x : c\ell^{\zeta_p'} \leqslant |\delta_\ell \mathbf{u}(x)| \leqslant C\ell^{\zeta_p'}\}.$$

Here, the two exponents are in the same correspondence as before $h = \zeta_p'$. Indeed, we show that

$$(7) \qquad \langle 1_A \rangle \lesssim \ell^{3 - d_h} = \ell^{3 - D_p} = V_p.$$

For more detailed multifractal analysis of Hölder spectrum based on Besov scaling we refer to [Jaf97, Jaf00, Dri21].

In principle, an arbitrary smooth concave exponent ζ_p passing through the origin could be achieved by a velocity field as we demonstrate in some concrete examples in Subsections 2.4 and 2.6. However, in fully developed turbulence one might expect universal laws for ζ_p and intermittency dimensions D_p, d_h. Mathematically, such universality can be studied by analysing vector fields with randomized Fourier

coefficients. To this end, in Subsection 2.7 we compute the expected spectrum for a random vector field normalized in accordance with the Kolmogorov $\frac{4}{5}$th law: $\mathbb{E}\zeta_3 = 1$. We derive the following bounds for the expected value of ζ_p:

$$\frac{p}{3} + \frac{-\ln\sqrt{\pi} + p\ln 2 + \ln\Gamma((p+1)/2)}{\ln\ell} \leqslant \mathbb{E}\zeta_p \leqslant p\left(\frac{1}{3} - \frac{2\ln 2}{3\ln\ell}\right).$$

Intermittency dimension D_p for the lower bound is computed to be

$$D_p = 3 - \frac{-\ln\sqrt{\pi} + \ln\Gamma((p+1)/2) - \frac{p}{2}\psi((p+1)/2)}{\ln\ell},$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the polygamma function. While randomized fields may exhibit intermittency at finite scales, the classical K41 spectrum holds in the limit as $\ell \to 0$:

$$\lim_{\ell\to 0}\mathbb{E}\zeta_p = \frac{p}{3},$$

confirming the heuristics that frequency randomized fields loose concentration and eddies start occupying the whole space. In the particular case of $p = 2$, the scaling of the second order structure function becomes $S_2(\ell) \to \ell^{\frac{2}{3}}$, as $\ell \to 0$. This can be translated directly to asymptotic behavior of the energy spectrum $E(\kappa) \to \frac{c_0}{\kappa^{5/3}}$, as $\kappa \to \infty$, recovering the classical law of 5/3-rds. Relationship between the energy spectrum and second structure function will be established for any scaling laws in Proposition 4.1.

In Section 3 we give an information-theoretic analysis of the field based on the introduce volumetric parameters. The main main idea of this study is to establish

1) how much information about the vector field $\delta_\ell\mathbf{u}$ is contained in sets of probability measure $V_{p,q}$ and V_p;

2) how much information about the vector field $\delta_\ell\mathbf{u}$ is contained in the corresponding sets $A_{q,p}$ and A_p themselves, while their probability measures being asymptotically subordinate to the respective volumes in a manner similar to (7).

One surprising result of this study shows that the dyadic volume factors of type $V_{q,q/2}$ collect most information about the L^q-source $|\delta_\ell\mathbf{u}|^q/\langle|\delta_\ell\mathbf{u}|^p\rangle$ among all the factors $V_{q,p}$, $p < q$. A particular case is given by the classical flatness factor $\mathcal{F} = V_{4,2}^{-1}$. At the same time, the 1-parameter families V_p and A_p's collect information about the entropy source $F\ln F$, where $F = |\delta_\ell\mathbf{u}|^p/\langle|\delta_\ell\mathbf{u}|^p\rangle$.

What constitutes and defines the active part of the field $\delta_\ell\mathbf{u}$ can be expressed via an explicit family of threshold speeds:

$$(8) \qquad s_{q,p} = \frac{\langle|\delta_\ell\mathbf{u}|^q\rangle^{\frac{1}{q-p}}}{\langle|\delta_\ell\mathbf{u}|^p\rangle^{\frac{1}{q-p}}},$$

and their limiting 1-parameter counterpart

$$(9) \qquad \lim_{q\to p} s_{q,p} = s_p = \exp\left\{\frac{\langle|\delta_\ell\mathbf{u}|^p\ln|\delta_\ell\mathbf{u}|\rangle}{\langle|\delta_\ell\mathbf{u}|^p\rangle}\right\}.$$

Information about these parameters can be gathered experimentally. Results of this research were reported in [NLK$^+$19].

2. Intermittency and multifractal spectrum

2.1. **Intermittency dimensions D_p, d_h and scaling exponents ζ_p.** For each scale ℓ and $p \in \mathbb{R}$ we define

$$\zeta_p := \zeta_p(\ell) = \log_\ell\langle|\delta_\ell\mathbf{u}|^p\rangle.$$

In other words, ζ_p is the exponent of the pth order structure function,

$$(10) \qquad S_p(\ell) = \langle|\delta_\ell\mathbf{u}|^p\rangle = \ell^{\zeta_p}, \qquad p \in \mathbb{R}.$$

Note that all the exponents pass through the origin,

$$\zeta_0 = 0.$$

Although in general ζ's depend on ℓ, we will suppress it to simplify the notation.

Let us define the domain of ζ_p:

$$p_{\min} = \inf\{p \leqslant 0 : \zeta_p \text{ is finite}\}, \qquad p_{\max} = \sup\{p \geqslant 0 : \zeta_p \text{ is finite}\}.$$

Note that the endpoints $p_{\min}$ or $p_{\max}$ might not be be in the domain of ζ_p.

Lemma 2.1. *For $\ell \leqslant 1$, as a function of p, ζ_p is concave and smooth on $(p_{\min}, p_{\max})$ with*

(11)
$$\zeta_p' = \frac{\langle |\delta_\ell \mathbf{u}|^p \log_\ell |\delta_\ell \mathbf{u}| \rangle}{\langle |\delta_\ell \mathbf{u}|^p \rangle}.$$

Proof. The smoothness follows directly from the definition. To show concavity, first note that ζ_p is trivially concave in p,

$$\zeta_{\theta p_1 + (1-\theta) p_2} \geqslant \theta \zeta_{p_1} + (1-\theta) \zeta_{p_2}, \quad 0 \leqslant \theta \leqslant 1,$$

by the Hölder inequality on each interval $[0, p_{\max})$ and $(p_{\min}, 0]$ separately. Combining with the fact that it is also differentiable at $p = 0$ shows that ζ_p is concave globally on the entire interval. $\qquad\square$

The basic idea of the Frisch-Parisi theory consists of relating the scaling exponents ζ_p to a range of parameters $d_h(\ell)$, called multifractal spectrum (MFR for short), that at a given scale ℓ measure the dimension of a set A_h where the field $\mathbf{u}$ achieves the Hölder exponent h. The relation between the MFR and the scaling exponents can be derived heuristically based on an incidence argument, see [Fri95] and is given by Legendre transform

(12)
$$d_h(\ell) = \inf_p (3 + hp - \zeta_p).$$

Here the infimum is taken over the domain of ζ_p. We will use (12) as the definition of $d_h(\ell)$ over domain

$$[h_{\min}, h_{\max}] = [\inf_p \zeta_p', \sup_p \zeta_p'],$$

where again the infimum and supremum are taken over the domain of ζ_p.

We can in fact justify (12) rigorously defining the following active region capturing specific Hölder regularity of order h at a specific scale ℓ

(13)
$$A_h = \{x : c\ell^h \leqslant |\delta_\ell \mathbf{u}(x)| \leqslant C\ell^h\}.$$

Here $C, c > 0$ are fixed constants. Then, by Chebyshev's inequality, for $p \geqslant 0$ we obtain

(14)
$$\langle 1_{A_h} \rangle \leqslant c^{-p} \ell^{-hp} \langle |\delta_\ell \mathbf{u}|^p \rangle,$$

and for $p < 0$, using the upper bound defining A_h,

(15)
$$\langle 1_{A_h} \rangle \leqslant C^{-p} \ell^{-hp} \langle |\delta_\ell \mathbf{u}|^p \rangle.$$

Combining with the very definition (10) we obtain

$$\langle 1_{A_h} \rangle \lesssim \ell^{-hp + \zeta_p}, \qquad p \in \mathbb{R}.$$

Since ℓ is small, the optimal bound is obtained at the point of the minimal exponent resulting in

(16)
$$\langle 1_{A_h} \rangle \lesssim \ell^{-\inf_p (hp - \zeta_p)} = \ell^{3 - d_h(\ell)}.$$

This rigorous inequality demonstrates that the asymptotic value $d_h = \lim_{\ell \to 0} d_h(\ell)$ captures the fractal dimension of the set of Hölder regularity h.

Knowing the exponents ζ_p or, equivalently, structure functions S_p, allows us to compute the MFR according to (12). On the other hand, assuming that we knew the MFR in the first place, its practical use can come from inverting formula (12) over the domain of ζ_p

(17)
$$\zeta_p = \inf_h (3 + ph - d_h(\ell)),$$

where the infimum is taken over $[h_{\min}, h_{\max}]$. This gives a direct access to computing scaling exponents ζ_p from the multifractal spectrum. It is therefore highly desirable to define the dimensions $d_h(\ell)$ on the first place and in a way that is amenable to analysis.

To this end, we start by defining a family of quantities called volume factors. Let $-\infty \leqslant p, q \leqslant \infty$, $q \neq p$. We define the (q, p)-*volume factor* at scale ℓ as follows

$$(18) \qquad V_{q,p}(\ell) = \frac{\langle |\delta_\ell \mathbf{u}|^q \rangle^{\frac{p}{p-q}}}{\langle |\delta_\ell \mathbf{u}|^p \rangle^{\frac{q}{p-q}}}.$$

We will postpone the detailed study of these parameters till later sections. In short, the factors capture the probability measure of the set where "most" of the L^q-mass of the field is concentrated, see Lemma 3.2. As $\ell \to 0$ such "active" sets settle on a fractal of certain dimension, which is determined by the exponential rate of decay of the volume factors. So, we further define

$$(19) \qquad D_{q,p}(\ell) = 3 - \log_\ell V_{q,p}.$$

These dimensions can be directly expressed in terms of scaling exponents ζ_p's via

$$(20) \qquad D_{q,p}(\ell) = 3 - \frac{p\zeta_q - q\zeta_p}{p - q}.$$

Remark 2.2. A more illuminating role of the dimensions is seen if one rewrites its definition in the following form

$$(21) \qquad \langle |\delta_\ell \mathbf{u}|^p \rangle^{\frac{1}{p}} = \ell^{(3 - D_{p,q})\left(\frac{1}{p} - \frac{1}{q}\right)} \langle |\delta_\ell \mathbf{u}|^q \rangle^{\frac{1}{q}}.$$

This equality bears direct relation with the Bernstein equality for Littlewood-Paley projections of the field $\mathbf{u}$, see [Gra14]. In fact if we replace the velocity increments $\delta_\ell \mathbf{u}$ with the corresponding projections $\mathbf{u}_k$ to the dyadic shells in the Fourier space around a wavenumber $\lambda_k = 2^k / L$, where L is the characteristic length scale of the periodic domain $L\mathbb{T}^n$, then the classical Bernstein inequality states that for $p < q$ one has

$$\|\mathbf{u}_k\|_q \leqslant c\lambda_k^{3\left(\frac{1}{p} - \frac{1}{q}\right)} \|\mathbf{u}_k\|_p,$$

where $c > 0$ is an adimensional constant. At the same time, on the periodic domain $\|\mathbf{u}_k\|_q \geqslant c(L)\|\mathbf{u}_k\|_p$. So, there exists a parameter $D_{q,p}$ which gauges the level of saturation of the inequality:

$$(22) \qquad \|\mathbf{u}_k\|_q = \lambda_k^{(3 - D_{q,p})\left(\frac{1}{p} - \frac{1}{q}\right)} \|\mathbf{u}_k\|_p.$$

We can see that (22) is precisely (21) taken with the corresponding spacial scale $\ell = \lambda_k^{-1}$.

This definition of dimension based on the Bernstein inequality for the particular case $p = 2$, $q = 3$ was adopted by the authors in work [CS14a], which recovered the β-model of intermittency [FSN78], see also [Fri95]. We will come back to this point in Section 2.4 and present a similar interpretation based on the finite difference approach taken here.

Now, letting $q \to p$ we define a new 1-parameter family of volumes and corresponding dimensions:

$$(23) \qquad \begin{aligned} V_{q,p} &\to V_p = \langle |\delta_\ell \mathbf{u}|^p \rangle \exp\left\{ -\frac{\langle |\delta_\ell \mathbf{u}|^p \ln |\delta_\ell \mathbf{u}|^p \rangle}{\langle |\delta_\ell \mathbf{u}|^p \rangle} \right\} \\ D_{q,p} &\to D_p = 3 - \log_\ell \langle |\delta_\ell \mathbf{u}|^p \rangle + \frac{\langle |\delta_\ell \mathbf{u}|^p \log_\ell |\delta_\ell \mathbf{u}|^p \rangle}{\langle |\delta_\ell \mathbf{u}|^p \rangle}. \end{aligned}$$

Notice that

$$(24) \qquad V_p = \ell^{3 - D_p},$$

and by virtue of the formula (20) we obtain the following relationship between the new dimensions D_p and ζ_p's:

$$(25) \qquad D_p = 3 - \zeta_p + p\zeta_p'.$$

Recalling the classical fact that the infimum (12) occurs at a point p where $h = \zeta_p'$, we conclude that

$$D_p(\ell) = d_h(\ell), \quad h = \zeta_p'.$$

This formula shows the new and old dimensions are in natural correspondence via the monotone change of variables $h = \zeta_p'$. Consequently the range D_p covers the same full multifractal spectrum as p runs through the interval $[p_{\min}, p_{\max}]$, just as d_h does as h runs through its interval of possible values $[h_{\min}, h_{\max}]$, with a possible exclusion of the endpoints. Since ζ_p' is monotonely decreasing in p, the end-point values of $h = \zeta_p'$ are determined by

$$(26) \qquad\qquad h_{\min} = \zeta_{p_{\max}}' \qquad \text{and} \qquad h_{\max} = \zeta_{p_{\min}}',$$

where the values are to be interpreted as limits which always exist, even if infinite, by monotonicity.

Thanks to (24), the bound (16) on the active region A_h can now be expressed in terms of the volume V_p:

$$(27) \qquad\qquad \langle 1_{A_h} \rangle \lesssim \ell^{3-d_h} = \ell^{3-D_p} = V_p,$$

and hence we shall refer to V_p as an active volume. Also, the threshold ℓ^h determining the active region A_h can be expressed more explicitly using that $h = \zeta_p'$ and formula (11)

$$(28) \qquad\qquad \ell^h = \ell^{\frac{\langle |\delta_\ell \mathbf{u}|^p \log_\ell |\delta_\ell \mathbf{u}| \rangle}{\langle |\delta_\ell \mathbf{u}|^p \rangle}} = \exp \left\{ \frac{\langle |\delta_\ell \mathbf{u}|^p \ln |\delta_\ell \mathbf{u}| \rangle}{\langle |\delta_\ell \mathbf{u}|^p \rangle} \right\} =: s_p.$$

These quantities s_p determine another 1-parameter family of values which determine "active thresholds" intimately tied with the other volumetric quantities introduced above. We will address this systematically in Section 3.

By analogy with the relation (17) we can now use D_p as a new determining family of dimensions. In fact solving the simple ODE (25) for ζ_p we obtain, for all $p \in (p_{\min}, p_{\max})$,

$$(29) \qquad\qquad \zeta_p = p\zeta_0' + p \int_0^p \frac{D_s - 3}{s^2} \, ds.$$

So the relationship between ζ_p's and D_p's is purely differential, given by (25), (29), while the relationship between ζ_p's and the old dimensions d_h is variational (12), (17).

Formula (29) shows explicitly the intermittency correction to the classical linear profile expressed in terms of the new dimensions

$$(30) \qquad\qquad \mathcal{I}_p = p \int_0^p \frac{D_s - 3}{s^2} \, ds.$$

In particular, if $D_p \equiv 3$ as in the Kolmogorov regime, then $\mathcal{I}_p = 0$ and we recover the classical theory of non-intermittent turbulence.

Interestingly, while $D_p = D_{p,p}$ define only a particular subfamily dimensions $D_{q,p}$, the full 2-parameter family can be restored from its diagonal values D_p. Indeed, plugging (29) into (20) we obtain

$$(31) \qquad\qquad D_{q,p} = 3 + \frac{qp}{q-p} \int_p^q \frac{D_s - 3}{s^2} \, ds.$$

This can be further simplified if p and q have the same sign by splitting the integral:

$$(32) \qquad\qquad D_{q,p} = \frac{qp}{q-p} \int_p^q \frac{D_s}{s^2} \, ds, \quad qp > 0.$$

2.2. Functional properties of dimensions.

Now that we have defined the multifractal spectrum, it is desirable to understand the range of possible Hölder exponents $[h_{\min}, h_{\max}]$ and other graph properties of the dimensions.

First, directly from the definition it is clear that D_p is smooth on $(p_{\min}, p_{\max})$. A more explicit formula for D_p is given by

$$(33) \qquad\qquad D_p = 3 - \log_\ell \langle |\delta_\ell \mathbf{u}|^p \rangle + \frac{\langle |\delta_\ell \mathbf{u}|^p \log_\ell |\delta_\ell \mathbf{u}|^p \rangle}{\langle |\delta_\ell \mathbf{u}|^p \rangle}.$$

In particular, $D_0 = 3$. We also have
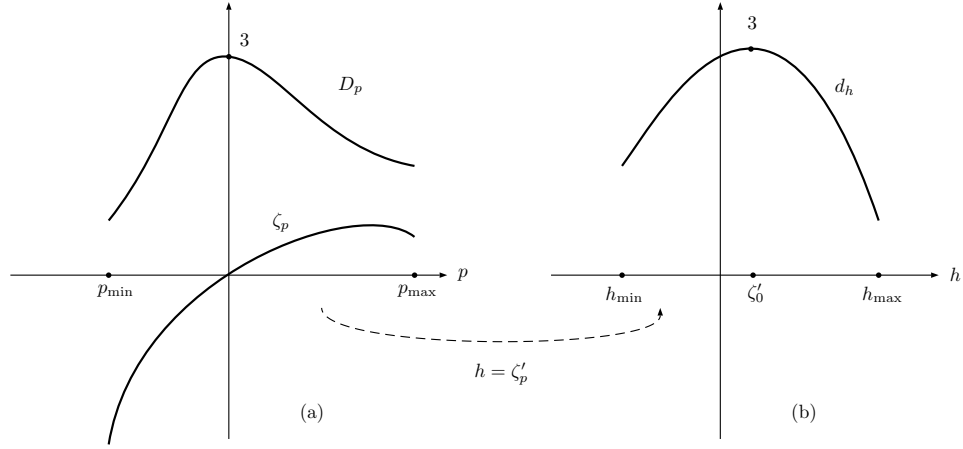
$$D_p' = p\zeta_p'',$$

FIGURE 1. (a) generic graphs of D_p and ζ_p; (b) generic graph of d_h. This also corresponds to case (1) of the end-point behavior.
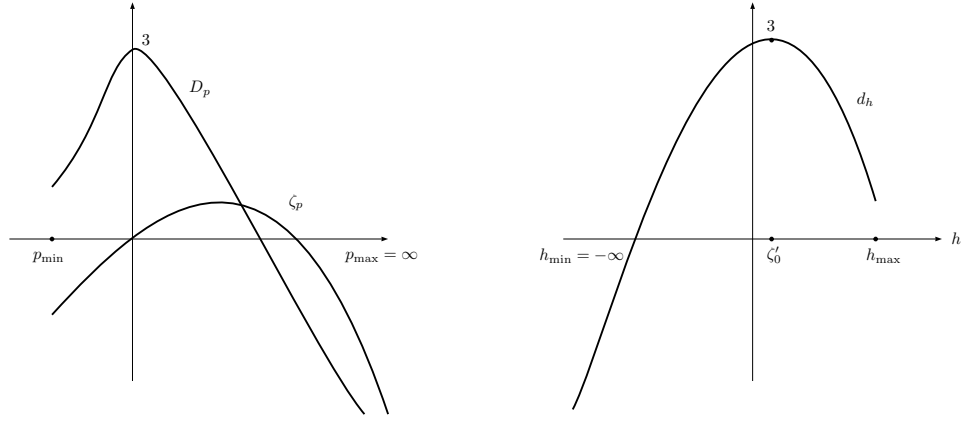

FIGURE 2. Case (2) of the end-point behavior.

which in view of concavity, $\zeta_p'' \leqslant 0$, implies that $D_p' \geqslant 0$ for $p \leqslant 0$, and $D_p' \leqslant 0$ for $p \geqslant 0$. Consequently, D_p attains its maximum at the origin.

As a function of p the dimensions D_p are generally not concave, see Figure 4. However, as a function of Hölder exponent h, the dimensions d_h are obviously concave from the general properties of the Legendre transform. In fact, if $p = p(h)$ is the point where the infimum in (12) is attained, which is the inverse of the decreasing function $h = \zeta_p'$, then $p(h)$ is decreasing as well. Moreover, we have

$$d_h' = p(h), \quad d_h'' = p'(h) = \frac{1}{\zeta_p''} \leqslant 0, \quad h \in (h_{\min}, h_{\max}),$$

but this might not hold at the endpoints $h_{\min}$ or $h_{\max}$ as we will see below. We also see that negative p's define the interval of decreasing behavior for d_h and positive values define the interval of the increase. In terms of h, this means that d_h is increasing on the interval $(h_{\min}, \zeta_0']$, picks at value $d_{\zeta_0'} = D_0 = 3$, and decreases on $[\zeta_0', h_{\max})$, see Figure 1.

Now we turn to the analysis of two endpoints $(p, h) = (p_{\max}, h_{\min})$ and $(p, h) = (p_{\min}, h_{\max})$. For each of them there are five possible scenarios. We will only describe them for $(p_{\max}, h_{\min})$ as for the other end point the corresponding cases can be obtained by interchanging min and max.

(1) $p_{\max}$ and $h_{\min}$ are finite. Then $\zeta_{p_{\max}}' = h_{\min}$, $\zeta_{p_{\max}}$, $d_{h_{\min}}$, and $d_{h_{\min}}' = p_{\max}$ are all finite, see Figure 1.

(2) $p_{\max} = \infty$ and $h_{\min} = -\infty$. In this case $\zeta_{p_{\max}}' = -\infty$, $\zeta_{p_{\max}} = -\infty$, and $d_{h_{\min}} = d_{h_{\min}}' = -\infty$, see Figure 2.
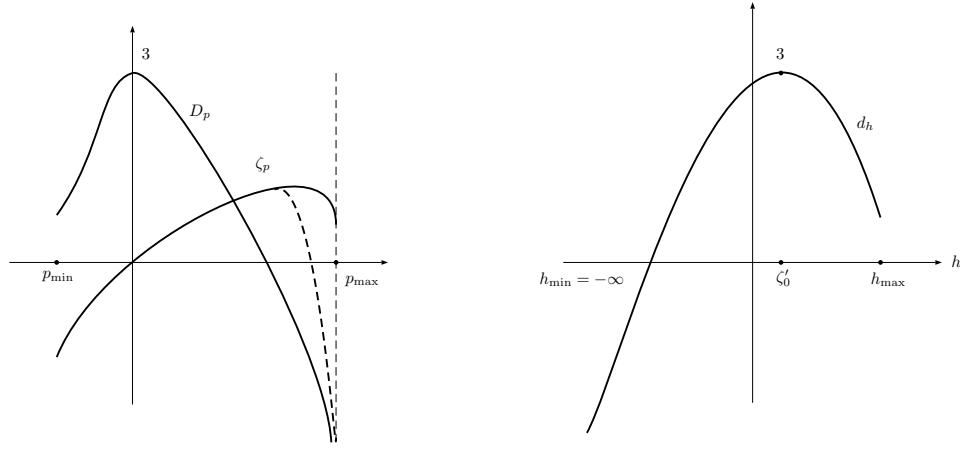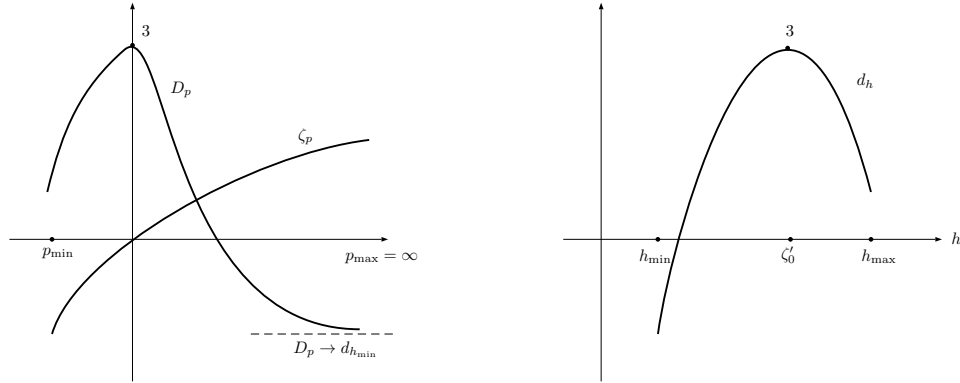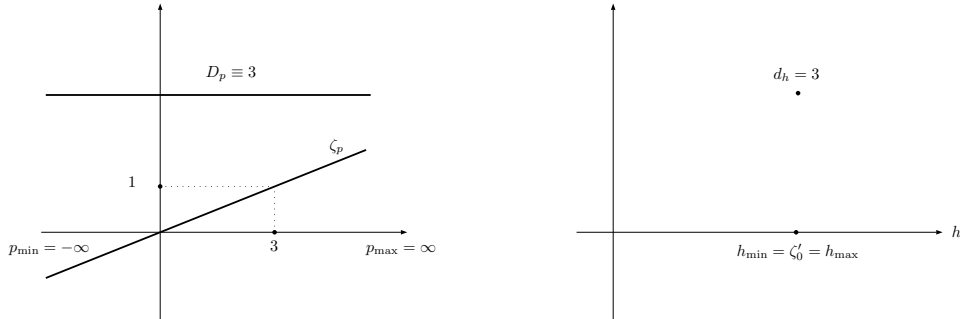
FIGURE 3. Case (3) of the end-point behavior.

FIGURE 4. Case (4) of the end-point behavior portraying the situation when $d_{h_{\min}}$ is finite. Note that in this case D_p has the horizontal asymptote at $d_{h_{\min}}$.

FIGURE 5. Case (5) represents the classical Kolmogorov-41 regime of turbulence.

(3) $p_{\max}$ is finite but $h_{\min} = -\infty$. In this case $\zeta'_{p_{\max}} = -\infty$, but $d'_{h_{\min}} = p_{\max}$ is finite. There are no restrictions on $\zeta_{p_{\max}} \in [-\infty, \infty)$ and $d_{h_{\min}} \in [-\infty, 3]$. If $p_{\max} > 0$, then $d_{h_{\min}} = -\infty$. See Figure 3.

(4) $p_{\max} = \infty$, $h_{\min}$ is finite, but $d'_{h_{\min}} = \infty$. In this case $\zeta'_{p_{\max}} = h_{\min}$ is finite. There are no restrictions on $\zeta_{p_{\max}} \in [-\infty, \infty]$ and $d_{h_{\min}} \in [-\infty, 3]$, see Figure 4.

(5) $p_{\max} = \infty$ but $h_{\min}$ and $d'_{h_{\min}}$ are finite. This implies K41 regime as we prove below. More precisely, $h_{\min} = h_{\max} = \zeta'_0$, $\zeta_p = \zeta'_0 p$, $D_p = 3$ for all $p \in \mathbb{R}$, and $d_{\zeta'_0} = 3$. Moreover, if $\zeta_3 = 1$, then $\zeta'_0 = 1/3$. See see Figure 5.

Remarkably, the so called *linearization phenomenon* occurs in case (4) as ζ_p converges to an asymptote $\zeta'_{p_{\max}} = h_{\min}$, which is consistent with some experimental observations and numerical computations [LBAC04, FGC+21].

Lemma 2.3. *Suppose that $p_{\max} = \infty$, $h_{\min} > -\infty$ and $d'_{h_{\min}} < \infty$. Then the conclusions of case (5) follow - the fluid is in the classical K41 regime.*

Proof. Indeed, in this case $d_{h_{\min}}$ has to be finite, and then the exponents ζ_p become linear in p, $\zeta_p = 3 + p h_{\min} - d_{h_{\min}}$ starting from $p \geqslant d'_{h_{\min}}$. This is because the infimum runs out of domain and is attained at the end-point $h = h_{\min}$. Also, $D_p = d_{h_{\min}}$ for all $p \geqslant d'_{h_{\min}}$ in this case. If such perfect linearization of ζ_p indeed occurs, then $\zeta''_p = 0$ for $p \geqslant d'_{h_{\min}}$. Then

$$\zeta''_p = \frac{1}{\ln \ell} \left[\frac{\langle |\delta_\ell \mathbf{u}|^p (\ln |\delta_\ell \mathbf{u}|)^2 \rangle}{\langle |\delta_\ell \mathbf{u}|^p \rangle} - \frac{\langle |\delta_\ell \mathbf{u}|^p \ln |\delta_\ell \mathbf{u}| \rangle^2}{\langle |\delta_\ell \mathbf{u}|^p \rangle^2} \right] = 0.$$

Hence

$$\langle |\delta_\ell \mathbf{u}|^{p/2} \ln |\delta_\ell \mathbf{u}| |\delta_\ell \mathbf{u}|^{p/2} \rangle^2 = \langle |\delta_\ell \mathbf{u}|^p (\ln |\delta_\ell \mathbf{u}|)^2 \rangle \langle |\delta_\ell \mathbf{u}|^p \rangle,$$

and the Hölder inequality becomes equality, in which case either $\delta_\ell \mathbf{u} \equiv 0$ or $|\delta_\ell \mathbf{u}|^{p/2} \ln |\delta_\ell \mathbf{u}|$ is proportional to $|\delta_\ell \mathbf{u}|^{p/2}$, and hence $|\delta_\ell \mathbf{u}| = constant$. Then $\zeta''_p = 0$ for all $p \in \mathbb{R}$, which immediately implies the conclusions of (5). $\square$

2.3. **Finer description of the Hölder spectrum.**
In addition to the defining formulas (26) it is useful to relate the Hólder exponents to the values of ζ_p's themselves.

Lemma 2.4. *We have the following general inequalities*

$$(34) \qquad h_{\min} \leqslant \frac{\zeta_q - \zeta_p}{q - p} \leqslant h_{\max}, \quad \forall p, q \in (p_{\min}, p_{\max}).$$

Furthermore, if $p_{\max} = \infty$ or $\zeta_{p_{\max}} = -\infty$, then

$$(35) \qquad h_{\min} = \lim_{p \to p_{\max}} \frac{\zeta_p}{p}.$$

Respectively, if $p_{\min} = -\infty$ or $\zeta_{p_{\min}} = -\infty$, then

$$(36) \qquad h_{\max} = \lim_{p \to p_{\min}} \frac{\zeta_p}{p}.$$

Proof. To prove (34) simply note that

$$\frac{\zeta_q - \zeta_p}{q - p} = \frac{1}{q - p} \int_p^q \zeta'_s \, \mathrm{d}s.$$

The integral, by monotonicity of ζ'_s, is bounded from above and below by $h_{\max}$ and $h_{\min}$, respectively.

To prove (35) by concavity and simple integration we have

$$(37) \qquad \zeta'_p \leqslant \frac{\zeta_p}{p} = \frac{1}{p} \int_0^p \zeta'_s \, \mathrm{d}s, \qquad \forall p \in [0, p_{\max}).$$

If $p_{\max} = \infty$, then passing to the limit in (37) we can see that both sides squeeze to the same limit $\zeta'_\infty = h_{\min}$. If $p_{\max} < \infty$ but $\zeta_{p_{\max}} = -\infty$, this means that both sides converge to $h_{\min} = -\infty$.

The proof at the opposite end goes entirely similar. $\square$

By taking $q = 0$ and $p = p_{\max}$ or $p = p_{\min}$, respectively, we in general obtain inequalities

$$h_{\min} \leqslant \lim_{p \to p_{\max}} \frac{\zeta_p}{p}, \qquad h_{\max} \geqslant \lim_{p \to p_{\min}} \frac{\zeta_p}{p}.$$

A pair of exact identities with dimensional corrections can be obtained by solving for $\zeta'_p = h$ in (25):

$$h_{\min} = \lim_{p \to p_{\max}} \left(\frac{\zeta_p}{p} - \frac{3 - D_p}{p} \right), \qquad h_{\max} = \lim_{p \to p_{\min}} \left(\frac{\zeta_p}{p} - \frac{3 - D_p}{p} \right),$$

One can use these to derive further finer properties of the end-point behavior of the spectra. For example, in case (4), i.e. if $p_{\max} = \infty$ and $h_{\min}$ is finite, there are two possibilities in terms of $d_{h_{\min}}$ being either finite or infinite. If it is finite, as depicted on Figure 4, then clearly D_p has to have a horizontal asymptote at $D_p \to d_{h_{\min}}$. However, even if $d_{h_{\min}} = \infty$, the graph of D_p would still have a zero slope at infinity. Indeed, in this case

$$h_{\min} = \lim_{p \to \infty} \left(\frac{\zeta_p}{p} - \frac{3 - D_p}{p} \right) = h_{\min} - \lim_{p \to \infty} \frac{D_p}{p},$$

and hence

$$\lim_{p \to \infty} \frac{D_p}{p} = 0.$$

In those cases where we do have the exact relations (35) and (36), the full Hölder spectrum of the field can also be recovered through the intermittency correction formula (29). We have

$$h_{\min} = \zeta_0' + \int_0^{p_{\max}} \frac{D_s - 3}{s^2} \, ds,$$

$$h_{\max} = \zeta_0' - \int_{p_{\min}}^0 \frac{D_s - 3}{s^2} \, ds.$$

Consequently, the full width of the spectrum is given by the total intermittency correction

$$h_{\max} - h_{\min} = \int_{p_{\min}}^{p_{\max}} \frac{3 - D_s}{s^2} \, ds.$$

Formulas (35) and (36) also allow to express the end-point Hölder exponents in terms of the maximum and minimum of the field itself.

Corollary 2.5. *If $p_{\max} = \infty$, then the minimal Hölder exponent is given by*

$$h_{\min} = \log_\ell \|\delta_\ell \mathbf{u}\|_\infty.$$

If $p_{\min} = -\infty$, then the maximal Hölder exponent is given by

$$h_{\max} = \log_\ell \min |\delta_\ell \mathbf{u}|.$$

In particular, if the solution belongs to the largest scaling invariant space $\mathbf{u} \in B_{\infty,\infty}^{-1}$, then $|\delta_\ell \mathbf{u}| \lesssim \ell^{-1}$, and so we have a limitation on the minimal Hölder exponent

$$h_{\min} \geqslant -1.$$

Looking at the other end, it is natural for exponent to stretch all the way to $h_{\max} = \infty$ because the flow at any given scale may have regions of infinite smoothness or even analyticity. The likelihood of this scenario is seen from the fact that fluctuations have mean zero, $\langle \delta_\ell \mathbf{u} \rangle = 0$. So, for example, for parallel shear flows it necessarily implies that $\delta_\ell \mathbf{u}$ must vanish somewhere, and hence $h_{\max} = \infty$ according to the corollary above.

2.4. **Under the Kolmogorov $\frac{4}{5}$th law: recovery of the β-model.** It is observed almost universally in experiments and simulations that the exponent of the third structure function is $\zeta_3 = 1$. This in part is evidenced by the Kolmogorov $\frac{4}{5}$th law which can be derived from the first principles, see [Fri95],

$$(38) \qquad S_3^\|(\ell) = \langle (\delta_\ell \mathbf{u} \cdot \ell/|\ell|)^3 \rangle = -\frac{4}{5} \varepsilon \ell,$$

where ε is the energy dissipation rate per unit mass. So, assuming that $\zeta_3 = 1$, formula (20) implies the famous relation between the structural exponents ζ_p's and the corresponding dimensions under the β-model formalism (compare with Frisch's [Fri95, (8.31)]):

$$(39) \qquad \zeta_p = \frac{p}{3} + (3 - D_{p,3}) \left(1 - \frac{p}{3} \right).$$

One particular case, $p = 2$, reads

$$(40) \qquad \zeta_2 = \frac{2}{3} + \frac{1}{3}(3 - D_{2,3}).$$

In the situation when the flow is self-similar, i.e. $\zeta_2(\ell) = \zeta_2$, we obtain the corresponding correction to the energy spectrum (compare with [Fri95, (8.32)]):

$$(41) \qquad\qquad E(\kappa) \sim \frac{1}{\kappa^{\frac{5}{3} + \frac{1}{3}(3 - D_{2,3})}}.$$

In fact, such connection between the scaling of the second order structure function and that of the energy spectrum is not entirely elementary, and we will prove this rigorously in Proposition 4.1 below.

The dependence of scaling exponent $h = \zeta_p'$ on p can be derived from (39),

$$(42) \qquad\qquad h = \frac{1}{3} - \frac{3 - D_{p,3}}{3} - \frac{3 - p}{3} D_{p,3}'.$$

This recovers formula [Fri95, (8.29)] obtained under the assumption of p-independent dimension. Note that the value $h = \frac{1}{3}$ is a part of this spectrum in view of Lemma 2.4.

To summarize, the entire set of predictions of the β-model becomes embedded into the formalism based on the two-parameter spectrum $\{D_{p,q}\}$ in the special case $q = 3$.

Our last observation concerns the content of the Hölder spectrum $I = [h_{\min}, h_{\max}]$. Dividing (39) by p and sending $p \to p_{\max}$ and $p_{\min}$ we obtain from Lemma 2.4,

$$(43) \qquad\qquad \begin{aligned} h_{\min} &\leqslant \frac{D_{p_{\max},3} - 2}{3}, \\ h_{\max} &\geqslant \frac{D_{p_{\min},3} - 2}{3}. \end{aligned}$$

It follows from Lemma 3.1 below that $D_{p,3}$ is a decreasing function of p. Hence, the entire range

$$\left\{ \frac{D_{p,3} - 2}{3} : p \in (p_{\min}, p_{\max}) \right\}$$

must be a part of the spectrum I.

Let us illustrate the findings of this section by an example of a mono-fractal turbulent field. Later in Section 2.6 we will upgrade it to multi-fractal construction.

We fix a scale ℓ and consider an arbitrary collection distribution of N cubes C_i, $i = 1, \ldots, N$, of linear size ℓ placed at a distance of ℓ from each other. Define

$$\mathbf{u} = \sum_{i=1}^{N} u_i \chi_{C_i},$$

where all $|u_i| = U_0$. The cubes occupy a volume of $N\ell^3 = \ell^{3-D}$, so by definition $D = -\log_\ell N \in [0, 3]$.

Let us compute the structure exponents now. We have

$$(44) \qquad\qquad \langle |\delta_\ell \mathbf{u}|^p \rangle = N U_0^p \ell^3 = U_0^p \ell^{3-D}.$$

On the other hand, under the Kolmogorov $\frac{4}{5}$th law, we must have

$$\langle |\delta_\ell \mathbf{u}|^3 \rangle \sim \varepsilon \ell.$$

Consequently, $U_0 = \varepsilon^{1/3} \ell^{(D-2)/3}$. Plugging into (44) we obtain

$$\langle |\delta_\ell \mathbf{u}|^p \rangle = \varepsilon^{p/3} \ell^{3 - D + p(D-2)/3}.$$

So, we arrive at the formula

$$\zeta_p = 3 - D + p(D - 2)/3, \qquad p > 0.$$

This is exactly the spectrum of the mono-fractal model given in (39). In fact, computing the dimensions we obtain $D_p = D$ for $p > 0$ directly from (25), and all $D_{q,p} = D$, $p, q > 0$. So, as we can see, the definitions recover the natural dimension of the field.

Computing the velocity thresholds from (28) we obtain all $s_p = U_0$, illustrating the fact that U_0 stands out as the active velocity of the eddies.

When $D < 3$, the graphs of D_p and ζ_p experience a jump discontinuity at $p = 0$, since as always $D_0 = 3$ and $\zeta_0 = 0$ by definition. This occurs due to idealization we took of the field itself picking only the active eddies, whereas in more realistic situation the field is more likely to be spread around at lower

levels. This would result in a rapid smooth transition from level 3 to D as far as D_p is concerned, and similarly for ζ_p. Consequently, the Hölder spectrum in our idealized example should be viewed as the ray $[h_{\min}, h_{\max}) = [\frac{D-2}{3}, \infty)$. The MFR dimensions satisfy $d_h = D$ for all $\frac{D-2}{3} \leqslant h < \infty$.

In Kolmogorov's regime $D = 3$ the exponent ζ_p is continuous at the origin and $h_{\min} = h_{\max} = 1/3$. Also $\zeta_0' = 1/3$ and consequently $d_{1/3} = D_0 = 3$. This case is illustrated on Figure 5.

For $D < 3$ we have $\zeta_0' = \infty$, so morally speaking we expect $d_\infty = 3$ even though d_∞ is not defined. This should be understood in the sense that $d_{h_{\max}}$ is close to 3 for a smooth approximation of ζ_p. To conclude, the predictions of our theory manifest themselves as expected, albeit in a somewhat degenerate form.

Finally, we remark that an intermittent mono-fractality is a feature of shocks, solutions to the Burgers' equation. For instance, in the one-dimensional case, the number of eddies is $N = 1$ at each scale ℓ, as $\delta_\ell \mathbf{u}$ is supported on an interval of length ℓ, and the argument above leads to

$$\zeta_p = 1, \qquad p > 0.$$

We can now compute the intermittency dimension, adjusting formula (25) to the one-dimensional case, $D_p = 1 - \zeta_p + p\zeta_p' = 0$. As expected, a Burgers' shock is highly intermittent, which can also be seen from the fact that its projections to dyadic frequency shells, which are of Dirichlet kernel type, saturate the Bernstein inequality.

2.5. **Energy flux and the third order exponent ζ_3.** The evidence for $\zeta_3 = 1$ which we postulated in the previous section can be seen in the context of deterministic weak solutions of the Euler equation

$$(45) \qquad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + f,$$

$$(46) \qquad \nabla \cdot \mathbf{u} = 0.$$

In [DR00] Duchon and Robert showed that for every weak solution $u \in L^3(\Omega_T)$ to the Euler equation one can associate a distribution $\mathcal{D}_\mathbf{u}$, representing the anomalous energy dissipation, such that

$$(47) \qquad \partial_t \left(\frac{1}{2}|\mathbf{u}|^2\right) + \operatorname{div}\left(\mathbf{u}\left(\frac{1}{2}|\mathbf{u}|^2 + p\right)\right) + \mathcal{D}_u = f \cdot \mathbf{u}.$$

In fact, $D_\mathbf{u}$ is the limit in the sense of distributions of

$$D_\mathbf{u}^\ell(x, t) = \frac{1}{4}\int \nabla\Psi^\ell(y) \cdot \delta_y\mathbf{u}|\delta_y\mathbf{u}|^2\, dy,$$

for any non-negative compactly supported mollifier $\Psi^\ell(x) = \ell^{-3}\Psi(x/\ell)$, where $\delta_y\mathbf{u} = \mathbf{u}(x+y, t) - \mathbf{u}(x, t)$. Choosing radially symmetric mollifier $\nabla\Psi^\ell(x) = \ell^{-4}\Phi(|x|/\ell)$ supported on $\alpha^{-1}\ell \leqslant r \leqslant \alpha\ell$ we compute

$$\varepsilon_\ell := \langle|\mathcal{D}_u^\ell|\rangle \leqslant \frac{1}{4\ell^4}\int_{\alpha^{-1}\ell}^{\alpha\ell} |\Phi(r/\ell)|4\pi r^2 \langle|\delta_r\mathbf{u}|^3\rangle\, dr$$
$$\sim \ell^{-1}\langle|\delta_r\mathbf{u}|^3\rangle_{r\sim\ell},$$

where heuristically ε_ℓ represents the energy flux through scale ℓ. Thus

$$\langle\zeta_3(r)\rangle_{r\sim\ell} \geqslant 1 + \log_\ell(c\varepsilon_\ell) \to 1,$$

as $\ell \to 0$, provided $0 < \lim_{\ell\to 0}\varepsilon_\ell < \infty$, as expected for turbulent flows.

On the other hand, a similar computation yields an upper bound for the longitudinal exponent, defined as

$$\zeta_3^\|(\ell) = \log_\ell\langle(\delta_\ell\mathbf{u} \cdot \ell/|\ell|)^3\rangle,$$

assuming some regularity of $\mathbf{u}$.

2.6. **A simple example of multifractality.** By analogy with the example presented in Section 2.4 we can construct a field whose multifractal spectrum approximates an arbitrary convex dimension d_h, and hence, approximates an arbitrary convex power law ζ_p.

In order not to make it exceedingly complicated we allow the field to vanish as before, and therefore we will adhere to the case of $p_{\min} = 0$. In other words, we assume that d_h is non-decreasing since $p = d'_h$. Let us also assume that the end-points of the spectrum are finite, i.e., $h_{\min}, h_{\max}, p_{\min} = 0, p_{\max} \in \mathbb{R}$. In other words, the slopes of the graphs of ζ_p and d_h are finite at the ends, and $d_{h_{\max}} = 3$.

Let us fix a sequence $h_{\min} = h_1 < h_2 < \cdots < h_K = h_{\max}$. Also, fix dimensions $\mathcal{D}_k = d_{h_k}$, which forms a monotonely increasing sequence. We now define a field $\mathbf{u}$, by considering K families of cubes $C_{i,k}$, $k = 1, \ldots, K$, $i = 1, \ldots, N_k$, or size ℓ separated by a distance of at least ℓ form each other. We assume $|u_{i,k}| = \ell^{h_k}$, and $N_k = \frac{1}{\ell^{\mathcal{D}_k}}$. Define

$$\mathbf{u} = \sum_{k=1}^{K} \sum_{i=1}^{N_k} u_{i,k} \chi_{C_{i,k}}.$$

Then

$$\langle |\delta_\ell \mathbf{u}|^p \rangle = \sum_{k=1}^{K} \ell^{3 + ph_k - \mathcal{D}_k} \sim \ell^{\zeta_p^K},$$

where

$$\zeta_p^K = \min_k (3 + ph_k - \mathcal{D}_k).$$

So the (smooth) power spectrum for the constructed vector field $\mathbf{u}$ is close to a polygon ζ_p^K with nodal points given by

$$p_k = \frac{\mathcal{D}_{k+1} - \mathcal{D}_k}{h_{k+1} - h_k}, \quad \zeta_{p_k}^K = 3 + \frac{\mathcal{D}_{k+1} h_k - \mathcal{D}_k h_{k+1}}{h_{k+1} - h_k}, \quad k = 1, \ldots K - 1,$$

and $p_K = 0$, $\zeta_{p_K} = 0$. The graph of ζ_p is inscribed under the polygon, whose edges are tangent to ζ_p at points $p = d'_{h_k}$.

Now we can use formula (25) to find the intermittency dimension D_p corresponding to the polygon ζ_p^K, which will approximate the intermittency dimension of the constructed field $\mathbf{u}$. The dimension function D_p is not well-defined at the nodal points p_k's, but otherwise is given by

$$D_p = \sum_{k=1}^{K} \mathcal{D}_k \chi_{(p_{k+1}, p_k)}(p).$$

As we let $\ell \to 0$, these fields will provide arbitrarily close approximation to the given data demonstrating that at least formally there is no functional restrictions on the possible multifractal spectra.

2.7. **Intermittent spectrum of a random field.** In this section we compute the expected spectrum of a vector field with randomized Fourier coefficients, and show that in the limit of vanishing scale $\ell \to 0$ the statistical laws of such a field approach the classical K41 prediction. Our computation are explicit enough and it allows to read off intermittency corrections at any finite scale.

Let us assume that the fluid domain is the torus $\mathbb{T}^3$. Let us fix a base field

$$\mathbf{u}_0 = \sum_{k \in \mathbb{Z}^3} u_k e^{ik \cdot x}.$$

We will assume that $\mathbf{u}_0$ is isotropic, i.e., the L^p norms of the velocity displacement do not depend on the direction of the displacement, and hence the p-th order structure function can be expressed as

$$(48) \qquad \langle |\delta_\ell \mathbf{u}_0|^p \rangle = \fint_{\mathbb{T}^3} |\mathbf{u}_0(x + \vec{\ell}) - \mathbf{u}_0(x)|^p \, dx,$$

for any vector $\vec{\ell}$ with $|\vec{\ell}| = \ell$. Such fields are expected and observed in turbulent flows, and mathematical examples are abundant. For instance, one can consider any radial field $\mathbf{u}_0 = \mathbf{u}_0(|x|)$ supported on a coordinate chart of the torus, or a combination of such disjoint fields separated by a distance of at least ℓ.

Based on the idea that in a fully turbulent flow Fourier coefficients of a field may point in random directions, we fix a set of independent identically distributed mean-zero Rademacher random variables $\{\theta_k\}_{k\in\mathbb{Z}_+}$, $\theta_k = \pm 1$, where

$$\mathbb{Z}_+^3 = \{(n_1, n_2, n_3) : n_1 > 0\} \cup \{(0, n_2, n_3) : n_2 > 0\} \cup \{(0, 0, n_3) : n_3 > 0\}.$$

We extend it to $\mathbb{Z}_-^3 = \mathbb{Z}^3 \setminus (\mathbb{Z}_+^3 \cup \{0\})$, by $\theta_{-k} = \theta_k$. Define the random field by

$$\mathbf{u} = \sum_{k\in\mathbb{Z}^3} u_k \theta_k e^{ik\cdot x},$$

where $u_{-k} = \bar{u}_k$ and $u_0 = 0$ are fixed. Note that $\mathbf{u}$ is no longer isotropic, however, for the 2nd order structure function we still have (48) because the L^2-norms are not random by Parseval's identity. In other words, the angle averaging in $\langle|\delta_\ell\mathbf{u}|^2\rangle$ can be omitted. Naturally, as before, we will use ℓ for both the displacement vector and its magnitude.

Given an isotropic vector filed, we decompose it as

$$\delta_\ell\mathbf{u} = \sum_{k\in\mathbb{Z}^3} u_k \theta_k e^{ik\cdot x}(e^{ik\cdot\ell} - 1) = \delta_\ell\mathbf{u}^+ + \delta_\ell\mathbf{u}^-,$$

where $\delta_\ell\mathbf{u}^\pm$ contains modes from $\mathbb{Z}_\pm^3$, respectively.

Let us compute expected value of the p-th order structure function. We will be primarily interested in the case where p is large, so we assume that $p \geqslant 3$ (although the computation below can be modified for all $p > 0$). Note that

$$\langle|\delta_\ell\mathbf{u}^-|^p\rangle = \langle|\overline{\delta_\ell\mathbf{u}^+}|^p\rangle = \langle|\delta_\ell\mathbf{u}^+|^p\rangle.$$

Hence, by the Hölder inequality,

$$\langle|\delta_\ell\mathbf{u}|^p\rangle = \langle|\delta_\ell\mathbf{u}^+ + \delta_\ell\mathbf{u}^-|^p\rangle \leqslant \langle(|\delta_\ell\mathbf{u}^+| + |\delta_\ell\mathbf{u}^-|)^p\rangle \leqslant 2^{p-1}\langle(|\delta_\ell\mathbf{u}^+|^p + |\delta_\ell\mathbf{u}^-|^p)\rangle = 2^p\langle|\delta_\ell\mathbf{u}^+|^p\rangle.$$

Recall that each $\delta_\ell\mathbf{u}^\pm$ is randomized by independent variables. According to the classical Khintchine inequality,

$$\mathbb{E}\langle|\delta_\ell\mathbf{u}|^p\rangle \leqslant 2^p\langle\mathbb{E}|\delta_\ell\mathbf{u}^+|^p\rangle$$

$$\leqslant 2^p B_p \left\langle \left(\sum_{k\in\mathbb{Z}_+^3} |u_k|^2 |e^{ik\cdot\ell} - 1|^2\right)^{p/2}\right\rangle_{\text{angle}}$$

$$= 2^p B_p \left\langle \left(\frac{1}{2}\sum_{k\in\mathbb{Z}^3} |u_k|^2 |e^{ik\cdot\ell} - 1|^2\right)^{p/2}\right\rangle_{\text{angle}}$$

$$= 2^{p/2} B_p \langle\langle|\delta_\ell\mathbf{u}|^2\rangle^{p/2}\rangle_{\text{angle}} = 2^{p/2} C_p \langle\langle|\delta_\ell\mathbf{u}_0|^2\rangle^{p/2}\rangle_{\text{angle}}$$

$$= 2^{p/2} B_p \langle|\delta_\ell\mathbf{u}_0|^2\rangle^{p/2} = 2^{p/2} C_p \langle|\delta_\ell\mathbf{u}|^2\rangle^{p/2},$$

where

$$B_p = 2^{p/2}\frac{1}{\sqrt{\pi}}\Gamma((p+1)/2).$$

So,

$$\mathbb{E}\langle|\delta_\ell\mathbf{u}|^p\rangle \leqslant C_p\langle|\delta_\ell\mathbf{u}|^2\rangle^{p/2}, \qquad C_p = 2^{p/2} B_p.$$

Since ζ_2 is not random, we have

$$\langle|\delta_\ell\mathbf{u}|^2\rangle^{p/2} = \ell^{\frac{p}{2}\zeta_2} = \ell^{\frac{p}{2}\mathbb{E}\zeta_2}.$$

At the same time, by the Jensen inequality, since ℓ^x is convex, we have

$$\mathbb{E}\langle|\delta_\ell\mathbf{u}|^p\rangle = \mathbb{E}\ell^{\zeta_p} \geqslant \ell^{\mathbb{E}\zeta_p}.$$

Thus, we obtain the following bounds on the structure functions:

$$\ell^{\frac{p}{2}\mathbb{E}\zeta_2} \leqslant \ell^{\mathbb{E}\zeta_p} \leqslant C_p\ell^{\frac{p}{2}\mathbb{E}\zeta_2}.$$
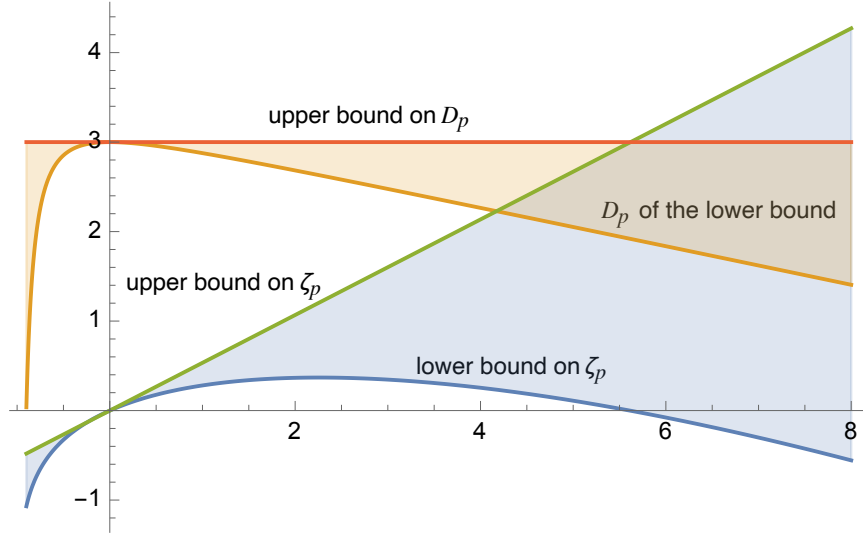
FIGURE 6. Graphs of the upper and lower bounds on ζ_p (51), upper bound on D_p, $D_p \leqslant 3$, and D_p computed for the lower bound on ζ_p (52) with $\ell = 0.1$.

Here, the left hand side follows by concavity. Potentiating,

$$(49) \qquad \frac{p}{2}\mathbb{E}\zeta_2 \geqslant \mathbb{E}\zeta_p \geqslant \frac{p}{2}\mathbb{E}\zeta_2 + \log_\ell C_p.$$

So, in the limit as $\ell \to 0$ we obtain

$$\mathbb{E}\zeta_p = \frac{p}{2}\mathbb{E}\zeta_2.$$

If the original field is normalized in accordance with the Kolmogorov's $\frac{4}{5}$th law, then $\mathbb{E}\zeta_3 = 1$, and we obtain the classical K41 spectrum

$$\lim_{\ell \to 0} \mathbb{E}\zeta_p = \frac{p}{3}.$$

Going back to (49) we can read off intermittency corrections at each scale quite explicitly:

$$(50) \qquad \frac{p}{2}\mathbb{E}\zeta_2 \geqslant \mathbb{E}\zeta_p \geqslant \frac{p}{2}\mathbb{E}\zeta_2 + \frac{-\ln\sqrt{\pi} + p\ln 2 + \ln\Gamma((p+1)/2)}{\ln\ell}.$$

Again, under the Kolmogorov normalization $\mathbb{E}\zeta_3 = 1$, computing (50) at $p = 3$ we obtain the estimates

$$\frac{2}{3} - \frac{4\ln 2}{3\ln\ell} \geqslant \mathbb{E}\zeta_2 \geqslant \frac{2}{3}.$$

Plugging back into (50) we obtain

$$(51) \qquad p\left(\frac{1}{3} - \frac{2\ln 2}{3\ln\ell}\right) \geqslant \mathbb{E}\zeta_p \geqslant \frac{p}{3} + \frac{-\ln\sqrt{\pi} + p\ln 2 + \ln\Gamma((p+1)/2)}{\ln\ell}.$$

Figure 6 shows the graphs of the upper and lower bounds on $\mathbb{E}\zeta_p$ in (51), as well as the intermittency dimension D_p for the lower bound in (51) computed according to (25)

$$(52) \qquad D_p = 3 - \frac{-\ln\sqrt{\pi} + \ln\Gamma((p+1)/2) - \frac{p}{2}\psi((p+1)/2)}{\ln\ell},$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the polygamma function. The smallest Hölder exponent in this case is $h_{\min} = -\infty$. We also see that the intermittency correction term

$$(53) \qquad I_p(\ell) = \frac{-\ln\sqrt{\pi} + p\ln 2 + \ln\Gamma((p+1)/2)}{\ln\ell}$$

becomes more prominent as p increases with ℓ being fixed, and for any fixed p it vanishes as $\ell \to 0$ pushing the exponent to the Kolmogorov regime $\mathbb{E}\zeta_p \to \frac{p}{3}$.

3. Volume factors, active velocities and regions

This section is devoted to analytical scrutiny of the newly introduced concepts. The main purpose here will be to demonstrate that the volume factors, dimensions and the associated thresholds define physically meaningful notions that extract the right information from the field $\mathbf{u}$ relevant in multifractal turbulence.

The actual origins of the field will not play a role here. We will simply fix a function f which belongs to a range of L^p-spaces defined over a probability measure space (Ω, μ). The averages $\langle \cdot \rangle$ mean the usual integration over Ω:

$$(54) \qquad \langle f \rangle = \int_\Omega f(\omega) \, \mathrm{d}\mu(\omega).$$

3.1. **Volume factors and dimensions.** We start with formalities of the active volume/region theory. Let $-\infty \leqslant p, q \leqslant \infty$, $q \neq p$. We define the (q,p)-*volume factor* of f as follows

$$(55) \qquad V_{q,p} = \frac{\langle |f|^q \rangle^{\frac{p}{p-q}}}{\langle |f|^p \rangle^{\frac{q}{p-q}}}.$$

Here are a few properties of volume factors that are easy to verify:

(V1) $V_{q,p}$ is adimensional;
(V2) symmetric: $V_{q,p} = V_{p,q}$;
(V3) homogeneous: $V_{q,p}(f) = V_{p,q}(\lambda f)$;
(V4) $V_{q,p} \leqslant 1$ if $pq \geqslant 0$, and $V_{q,p} \geqslant 1$ if $pq \leqslant 0$. Also, $V_{q,0} = V_{0,p} = 1$.
(V5) log-convex: for any triple $p_1 < p_2 < p_3$, and any q,

$$(56) \qquad \begin{aligned} V_{q,p_2} &\leqslant (V_{q,p_1})^{\frac{q-p_1}{q-p_2}\frac{p_3-p_2}{p_3-p_1}} (V_{q,p_3})^{\frac{q-p_3}{q-p_2}\frac{p_2-p_1}{p_3-p_1}}, \quad q > p_2 \\ V_{q,p_2} &\geqslant (V_{q,p_1})^{\frac{q-p_1}{q-p_2}\frac{p_3-p_2}{p_3-p_1}} (V_{q,p_3})^{\frac{q-p_3}{q-p_2}\frac{p_2-p_1}{p_3-p_1}}, \quad q < p_2. \end{aligned}$$

Let us simply notice that (V5) is a consequence of the interpolation inequality,

$$(57) \qquad \langle |f|^{p_2} \rangle \leqslant \langle |f|^{p_1} \rangle^{\frac{p_3-p_2}{p_3-p_1}} \langle |f|^{p_3} \rangle^{\frac{p_2-p_1}{p_3-p_1}}.$$

Slightly less trivial is the following property.

Lemma 3.1. *The volume-factors obey the following monotonicity properties:*

$$(V6) \qquad \partial_q V_{q,p} \leqslant 0 \text{ if } p > 0, \text{ and } \partial_q V_{q,p} \geqslant 0 \text{ if } p \leqslant 0.$$

Proof. If $q > p$, then we use the following form of $V_{q,p}$:

$$V_{q,p} = \langle |f|^p \rangle \left(\frac{1}{\left\langle \frac{|f|^p}{\langle |f|^p \rangle} |f|^{q-p} \right\rangle^{\frac{1}{q-p}}} \right)^p$$

Clearly the expression on the bottom represents a L^{q-p} norm with respect to the normalized measure $\frac{|f|^p}{\langle |f|^p \rangle} \, \mathrm{d}\mu$. Hence, the expression is increasing if $p > 0$ and decreasing if $p \leqslant 0$. For $q < p$ we first observe that

$$V_{q,p}^\varepsilon = \frac{\langle |f|^q \chi_{|f|>\varepsilon} \rangle^{\frac{p}{p-q}}}{\langle |f|^p \rangle^{\frac{q}{p-q}}} \to V_{q,p}.$$

But

$$V_{q,p}^\varepsilon = \langle |f|^p \rangle \left(\left\langle \frac{|f|^p}{\langle |f|^p \rangle} \left(\frac{\chi_{|f|>\varepsilon}}{|f|} \right)^{p-q} \right\rangle^{\frac{1}{p-q}} \right)^p.$$

By the same token $V_{q,p}^\varepsilon$ is decreasing in q if $p > 0$ and increasing if $p < 0$, and consequently so is $V_{q,p}$. This proves (V6). $\qquad \square$

We note that by the symmetry we have similar monotonicity in p. This makes $(0,0)$ a saddle point for the volume factors.

For a fixed scale $\ell < 1$, we also define the corresponding two-parameter family of dimensions

$$(58) \qquad\qquad D_{q,p} = 3 - \log_\ell V_{q,p}.$$

The log-convexity of expressed in (56) translates into the conventional convexity for the dimensions:

$$
\begin{aligned}
D_{q,p_2} &\leqslant \lambda_1 D_{q,p_1} + \lambda_2 D_{q,p_3}, \quad q > p_2 \\
D_{q,p_2} &\geqslant \lambda_1 D_{q,p_1} + \lambda_2 D_{q,p_3}, \quad q < p_2
\end{aligned}
$$

$$(59)$$

$$\lambda_1 = \frac{q - p_1}{q - p_2}\frac{p_3 - p_2}{p_3 - p_1}, \quad \lambda_2 = \frac{q - p_3}{q - p_2}\frac{p_2 - p_1}{p_3 - p_1}.$$

We recover the same monotonicity for $D_{q,p}$ as for volumes (V6), and from (V4) we obtain

$$(60) \qquad\qquad D_{0,p} = D_{q,0} = 3, \quad D_{q,p} \leqslant 3 \text{ if } pq \geqslant 0, D_{q,p} \geqslant 3 \text{ if } pq \leqslant 0.$$

3.2. **Concentration of information.** The basic analytical meaning of a volume factor is to give a measure of a set containing much of the information carried by the source f. More precisely, we have the following lemma.

Lemma 3.2. *Let* $0 \leqslant p < q \leqslant \infty$. *There exists a set* $A \subset \Omega$ *with* $\mu(A) = V_{q,p}$ *such that*

$$(61) \qquad\qquad (1 - c_{q,p}) \int_\Omega |f|^q \, d\mu \leqslant \int_A |f|^q \, d\mu,$$

where $c_{q,p} = \left(\frac{q-p}{q}\right)^{\frac{q-p}{q}} \left(\frac{p}{q}\right)^{\frac{p}{q}}$.

Proof. If $V_{q,p} = 1$, the statement is trivial. Suppose $V_{q,p} < 1$. Note that the function $\mu(\{|f| \geqslant \alpha\})$ is continuous from the left, and at a point of a jump the size of the jump is exactly $\mu(\{|f| = \alpha\})$. Hence, there exists an $\alpha \geqslant 0$ and a set $B \subset \{|f| = \alpha\}$ such that $A = \{|f| > \alpha\} \cup B$ has measure exactly $V_{q,p}$. By Chebyshev's inequality,

$$V_{q,p} \leqslant \frac{1}{\alpha^p} \int_A |f|^p d\mu.$$

Using this and the fact that $|f| \leqslant \alpha$ on $\Omega \backslash A$ we obtain

$$\langle |f|^q \chi_{\Omega\backslash A}\rangle \leqslant \alpha^{q-p}\langle |f|^p \chi_{\Omega\backslash A}\rangle \leqslant \frac{1}{V_{q,p}^{\frac{q-p}{p}}}\langle |f|^p \chi_A\rangle^{\frac{q-p}{p}}\langle |f|^p \chi_{\Omega\backslash A}\rangle \leqslant \langle |f|^q\rangle \frac{\langle |f|^p \chi_{\Omega\backslash A}\rangle^{\frac{p}{q}}\langle |f|^p \chi_A\rangle^{\frac{q-p}{p}}}{\langle |f|^p\rangle}$$

Note that the latter fraction is of the form $\theta^{\frac{p}{q}}(1-\theta)^{\frac{q-p}{p}}$, $\theta \in [0,1]$, which attains its maximum exactly at the value $c_{q,p}$. This proves the lemma. $\qquad\square$

Now there are several natural questions to ask.

First, for a fixed q, what is the most optimal exponent p that recovers most of the function? In other words, what is the smallest constant $c_{q,p}$? Elementary optimization yields, $c_{min} = \frac{1}{2}$, which is achieved when $p = \frac{q}{2}$. So, it implies that the volume factors will recover a half of the q-weight of the function at best, provided we choose $p = q/2$: there exists a set $A \subset \Omega$ with $\mu = V_{q,\frac{q}{2}}$ such that

$$(62) \qquad\qquad \frac{1}{2}\int_\Omega |f|^q \, d\mu \leqslant \int_A |f|^q \, d\mu.$$

It is worth noting that the classical flatness factor

$$\mathcal{F} = V_{4,2}^{-1} = \frac{\langle |f|^4\rangle}{\langle |f|^2\rangle^2}$$

emerges in this context as a particular case of an optimal concentration factor.

Second, we can see that on the other end letting $p \to q$ gives $c_{q,p} \to 1$. As a result, in this limit the information about concentration of the function f gets lost. However, the corresponding volume factors do not degenerate, and converge to something non-trivial, namely,

$$(63) \qquad V_p = \lim_{q \to p} V_{q,p} = \langle |f|^p \rangle \exp \left\{ -\frac{\langle |f|^p \ln |f|^p \rangle}{\langle |f|^p \rangle} \right\} = \exp \left\{ -\frac{\langle |f|^p \ln \frac{|f|^p}{\langle |f|^p \rangle} \rangle}{\langle |f|^p \rangle} \right\}.$$

Let us call them *p-volume factors*. The corresponding dimensional values defined by

$$(64) \qquad D_p := 3 - \log_\ell V_p = 3 - \log_\ell \langle |f|^p \rangle + \frac{\langle |f|^p \log_\ell |f|^p \rangle}{\langle |f|^p \rangle}.$$

have played the central role in the multifractal formalism we described in the previous section.

So, there arises a natural question: what kind of concentration not captured by Lemma 3.2 do these factors represent? The short answer – it is the concentration of the entropy of the renormalized distribution $F = \frac{|f|^p}{\langle |f|^p \rangle}$.

To start, let us define the entropy $H = \langle F \ln F \rangle$, and note the following simple formula

$$(65) \qquad V_p = e^{-H}.$$

Remark 3.3. As a side remark we note that V_p is directly related to the classical Shannon information: $\mathcal{N}(F) = \frac{1}{2\pi e} \exp \left\{ -\frac{2}{3} H \right\}$ via the power law:

$$\frac{1}{2\pi e} V_p^{\frac{2}{3}}(f) = \mathcal{N}(F),$$

see [Vil03] for more details.

The entropy H itself measures how close function f is to a constant via the classical Csiszár-Kullback inequality, see [Vil03]:

$$\frac{1}{2} \langle |F - 1| \rangle^2 \leqslant \langle F \ln F \rangle \leqslant \langle |F - 1|^2 \rangle.$$

It terms of f itself we obtain

$$(66) \qquad \frac{1}{2} \left\langle \left| \frac{|f|^p}{\langle |f|^p \rangle} - 1 \right| \right\rangle^2 \leqslant -\ln V_p \leqslant \left\langle \left| \frac{|f|^p}{\langle |f|^p \rangle} - 1 \right|^2 \right\rangle.$$

So, the closer V_p is to 1, the less concentrated the function f is, i.e. the more uniform it becomes.

Closer to the smaller range, however, the p-factors regulate concentration of the entropic density $F \ln F$. We will see this expressed in the following two lemmas that establish concentration in two different ways. First, concentration in a weak sense states that we have to increase the allowed volume slightly to $V^{1-\varepsilon}$ in order to achieve concentration up to the proportion ε. Although weak, this result works under no restrictions on H. And second, strong concentration states that $F \ln F$ indeed concentrates on a volume V_p but under an upper cap on the size of the entropies, $H \leqslant H_0$.

Both results hold for a general probability density F, its entropy $H = \langle F \ln F \rangle$, and volumetric factor $V = e^{-H}$.

Lemma 3.4 (Weak concentration). *For any probability distribution F, and any $0 < \varepsilon < 1$ there exists a set $A \subset \Omega$ with $\mu(A) = V^{1-\varepsilon}$ such that*

$$(67) \qquad \varepsilon H \leqslant \int_A F \ln F \, d\mu.$$

Proof. We start as in the proof of Lemma 3.2 and find $A \subset \{F \geqslant \alpha\}$ such that $\mu(A) = V^{1-\varepsilon}$. Then

$$\langle F \ln F \chi_{\Omega \setminus A} \rangle \leqslant \langle F \chi_{\Omega \setminus A} \rangle \ln \frac{\langle F \chi_A \rangle}{V^{1-\varepsilon}} = \langle F \chi_{\Omega \setminus A} \rangle (\ln \langle F \chi_A \rangle + (1 - \varepsilon) \langle F \ln F \rangle).$$

We have $H = \langle F \ln F \rangle$, and hence the above is bounded by the maximum of the function

$$g(x) = x \ln(1 - x) + (1 - \varepsilon) H x, \quad x \in [0, 1],$$

which is bounded by $(1 - \varepsilon) H$. $\qquad \square$

In terms of f, the result reads as follows

$$(68) \qquad \varepsilon \int_\Omega |f|^p \ln \frac{|f|^p}{\langle |f|^p \rangle} \, d\mu \leqslant \int_A |f|^p \ln \frac{|f|^p}{\langle |f|^p \rangle} \, d\mu.$$

The same proof in fact implies the strong version and allows us to reach the original volume $\mu(A) = V$ provided we assume that the entropy has a priori known bound $H \leqslant H_0$. Then, we obtain concentration with the constant c_{H_0} given by

$$c_{H_0} = \sup_{x \in [0,1], H \leqslant H_0} \frac{1}{H} x \ln(1 - x) + x.$$

Lemma 3.5 (Strong concentration)**.** *For any $H_0 > 0$ and probability distribution F, with $H \leqslant H_0$ there exists a set $A \subset \Omega$ with $\mu(A) = V$ such that*

$$(69) \qquad (1 - c_{H_0})H \leqslant \int_A F \ln F \, d\mu.$$

Unfortunately in this case we lose information $c_{H_0} \to 1$, as $H_0 \to \infty$. An example showing that this worsening is indeed happening can be constructed as follows. Let $\{I_i\}_{i=1}^\infty$ be a family of disjoint intervals on $[0, 1]$ with $|I_i| = \frac{1}{2^i}$. Define $F = \sum_{i=1}^n \frac{2^i}{n} \chi_{I_i}$. Then $\langle F \rangle = 1$ and

$$H = \frac{1}{n} \sum_{i=1}^n \ln \frac{2^i}{n} \sim n \ln 2 - \ln(n),$$

and $V \sim \frac{n}{2^n}$. In order to capture a set of this volume we need to take last $n - k$ intervals so that

$$|\{F \geqslant \alpha\}| = \sum_{i=k}^n \frac{1}{2^i} \sim \frac{1}{2^k} = \frac{n}{2^n}.$$

This implies $k = n - \log_2(n)$. But then

$$\int_A F \ln F \, dx = \frac{1}{n} \sum_{i=k}^n \ln \frac{2^i}{n} = \ln \frac{2^{\frac{k+\ldots+n}{n}}}{n^{\frac{n-k}{n}}} \lesssim \ln n,$$

which is an order smaller than H.

In summary, we have demonstrated that the volume factors represent a measure of concentration of the function f or its renormalized entropy $F \ln F$ in the case of 1-parameter factors. This motivates us to look further into the question of what hight levels of the function f, which already appeared in the proofs of this section, determine the threshold for its "most active" values.

3.3. **Active thresholds and regions.** Lemmas 3.2, 3.4, 3.5 provide little constructive information about the threshold α that define concentration sets A. In this section we will give a more physically relevant concept of an appropriate level α, called active threshold, which is similar in spirit to the volume factors. The upside of this approach is that such a definition will be computationally accessible, as we give a precise formula for α. A downside is that the active concentration set A will not have the exact same measure as $V_{q,p}$, but rather its constant multiple.

So, let us define an (q, p)-*active threshold* by

$$(70) \qquad s_{q,p} = \frac{\langle |f|^q \rangle^{\frac{1}{q-p}}}{\langle |f|^p \rangle^{\frac{1}{q-p}}}.$$

We recall that in the context of turbulence, $f = \delta_\ell \mathbf{u}$, and therefore $s_{q,p}$ depend on the scale. In [CS14a] these appeared in the special case of $p = 2$, $q = 3$, under the term of active speeds which refers to their physical unit.

By analogy with the previous section let us list some of the fundamental properties of active thresholds:

(s1) unit of $s_{q,p}$ is the same as the unit of f;
(s2) symmetric: $s_{q,p} = s_{p,q}$;
(s3) 1-homogeneous: $s_{q,p}(\lambda f) = \lambda s_{q,p}(f)$;

(s4) bound from above: $s_{q,p} \leqslant \|f\|_\infty$;

(s5) log-convexity: for any triple $p_1 < p_2 < p_3$, and any q,

(71)
$$s_{q,p_2} \leqslant (s_{q,p_1})^{\frac{q-p_1}{q-p_2}\frac{p_3-p_2}{p_3-p_1}} (s_{q,p_3})^{\frac{q-p_3}{q-p_2}\frac{p_2-p_1}{p_3-p_1}}, \quad q < p_2$$
$$s_{q,p_2} \geqslant (s_{q,p_1})^{\frac{q-p_1}{q-p_2}\frac{p_3-p_2}{p_3-p_1}} (s_{q,p_3})^{\frac{q-p_3}{q-p_2}\frac{p_2-p_1}{p_3-p_1}}, \quad q > p_2.$$

(s6) monotonicity: $\partial_q s_{q,p} \geqslant 0$ for any p.

(s7) volume-threshold-structure function relation:
$$s_{q,p}^p V_{q,p} = \langle |f|^p \rangle.$$

Note that (s6) is a consequence of (s7) and (V6).

Now to show relevance of the newly introduced thesholds to the concentration phenomena let us fix a family of adimensional constants $0 < \sigma_{q,p} < 1$, and define the (q, p)-*active region* by

(72)
$$A_{q,p} = \{|f| \geqslant \sigma_{q,p} s_{q,p}\}.$$

By Chebyshev's inequality, for $p > 0$, we readily obtain the bound
$$|A_{q,p}| \leqslant \frac{1}{\sigma_{q,p}^p s_{q,p}^p} \langle |f|^p \rangle = \frac{1}{\sigma_{q,p}^p} V_{q,p},$$

and in the case $p < q$,
$$\langle |f|^q \chi_{\Omega \setminus A_{q,p}} \rangle \leqslant \sigma_{q,p}^{q-p} \frac{\langle |f|^q \rangle}{\langle |f|^p \rangle} \langle |f|^p \chi_{\Omega \setminus A_{q,p}} \rangle \leqslant \sigma_{q,p}^{q-p} \langle |f|^q \rangle,$$

which implies the expected concentration property:

(73)
$$(1 - \sigma_{q,p}^{q-p}) \langle |f|^q \rangle \leqslant \langle |f|^q \chi_{A_{q,p}} \rangle.$$

Due to this inequality it is natural to set $\sigma_{q,p} = c_{q,p}^{\frac{1}{q-p}}$, which corresponds to the same level of concentration as the superlevel sets $\{|f| \geqslant \alpha\}$ in Lemma 3.2. Although such a choice of parameters may not be well-justified physically, it is certainly resonates mathematically with the prior result.

We now investigate the relevance of these active regions to the formalism of Hölder sets A_h we discussed in (13). To this end, we let $q \to p$. Once again, we observe the same phenomenon – the information about concentration of f gets lost since $\sigma_{q,p}^{q-p} \to 1$. However, the active thresholds converge to something non-trivial, namely

(74)
$$\lim_{q \to p} s_{q,p} = s_p = \exp\left\{ \frac{\langle |f|^p \ln|f| \rangle}{\langle |f|^p \rangle} \right\}.$$

A remarkable property of this 1-paramter family is that it stores information about the entire original 2-parameter family via the following restoration formula

(75)
$$s_{q,p} = \exp\left\{ \frac{1}{q-p} \int_p^q \ln s_r \, dr \right\}.$$

Concerning other properties the reader can readily check that s_p's inherit the same (s1)–(s4), (s6), while (s7) translates into

(76)
$$s_p^p V_p = \langle |f|^p \rangle.$$

Let us now go back to the physical interpretation of s_p's. First, let us observe that for $f = \delta_\ell \mathbf{u}$, $s_p = \ell^h$ are exact same values that appeared in (28) in our initial heuristic argument, which define the active regions A_h of Hölder regularity. There h is related to p via the usual transformation $h = \zeta_p'$. So, if we define A_p in the new terms parameterized by p

(77)
$$A_p = \{cs_p \leqslant |\delta_\ell \mathbf{u}| \leqslant Cs_p\},$$

then as before by Chebyshev's inequality and (76),
$$\mu(A_p) \lesssim \frac{\langle |\delta_\ell \mathbf{u}|^p \rangle}{s_p^p} = V_p = \ell^{3-D_p}.$$

This provides the direct analogue of (16) in terms of the volumetric quantities.

As in the last section there remains one last unsettled issue: since in the limit as $q \to p$ the concentration information of the sets $A_{q,p}$ deteriorates, we still would like to understand what kind of concentration do the limiting thresholds s_p capture? As previously we will find that it is responsible for concentration of entropic densities.

To that end, we extract the physical unit of s_p, equal that of f, upfront. Namely, if the physical unit of f is U_0 we rewrite the formula for s_p as follows:

$$(78) \qquad s_p = U_0 \exp \left\{ \frac{\langle |f|^p \ln \frac{|f|}{U_0} \rangle}{\langle |f|^p \rangle} \right\}.$$

Note that this is the exact same formula as appeared the original definition (74), and it holds for any U_0. We can view the ratio

$$\frac{s_p}{U_0} = \exp \left\{ \frac{\langle |f|^p \ln \frac{|f|}{U_0} \rangle}{\langle |f|^p \rangle} \right\}.$$

as an adimensional threshold value for an active velocity.

Lemma 3.6. *Let*

$$(79) \qquad A = \{ |f| \geqslant U_0^{1-c} s_p^c \},$$

where $0 < c < 1$. The function f concentrates on A in the following sense

$$(80) \qquad (1-c) \left\langle |f|^p \ln_+ \frac{|f|}{U_0} \right\rangle \leqslant \left\langle |f|^p \ln_+ \frac{|f|}{U_0} \chi_A \right\rangle.$$

Here we denote

$$\ln_+ = \begin{cases} \ln, & \text{if } \ln > 0 \\ 0, & \text{if } \ln = 0 \end{cases}$$

Proof. To see (80) note that on A we have

$$\frac{|f|}{U_0} \geqslant \left(\frac{s_p}{U_0} \right)^c.$$

Hence, on the complement we have

$$\left\langle |f|^p \ln_+ \frac{|f|}{U_0} \chi_{\Omega_T \setminus A} \right\rangle \leqslant c \left\langle |f|^p \frac{\langle |f|^p \ln \frac{|f|}{U_0} \rangle_+}{\langle |f|^p \rangle} \right\rangle = c \left\langle |f|^p \ln \frac{|f|}{U_0} \right\rangle_+ \leqslant c \left\langle |f|^p \ln_+ \frac{|f|}{U_0} \right\rangle.$$

This implies (80). $\qquad \square$

4. Energy spectrum

The purpose of this section is to provide a rigorous link between scaling properties of the second order structure function and the energy spectrum. To make it rigorous we define the energy spectrum classically as the Fourier transform of the properly defined correlation function.

So, let $\mathbf{u}$ be a velocity field. We define a 3D correlation function as follows

$$(81) \qquad \Gamma(\mathbf{y}) = \frac{1}{2} \int_{O(3)} \langle \mathbf{u}(\cdot + A\mathbf{y}) \cdot \mathbf{u}(\cdot) \rangle \, d\nu(A),$$

where $O(3)$ is the orthogonal group on $\mathbb{R}^3$, and $d\nu$ is the normalized Haar measure on it. Clearly, $\Gamma(0) = \mathcal{E} = \frac{1}{2} \langle |\mathbf{u}|^2 \rangle$ is the twice total energy. We will work under assumption that the correlation function decays sufficiently fast at infinity. It suffices to have

$$(82) \qquad |\nabla^k \Gamma(\mathbf{y})| \lesssim \frac{1}{1 + |\mathbf{y}|^{k-1}}, \quad k \leqslant 5.$$

We define the 3D energy spectrum by

(83)
$$E_{3D}(\boldsymbol{\kappa}) = \int_{\mathbb{R}^3} e^{-i\boldsymbol{\kappa}\cdot\mathbf{y}} \Gamma(\mathbf{y}) \, \mathrm{d}\mathbf{y}.$$

Using Parseval's identity we obtain

$$E_{3D}(\boldsymbol{\kappa}) = \frac{1}{2}\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{-i\boldsymbol{\kappa}\cdot\mathbf{y}} \int_{O(3)} |\hat{\mathbf{u}}(\boldsymbol{\xi})|^2 e^{iA\mathbf{y}\cdot\boldsymbol{\xi}} \, \mathrm{d}\boldsymbol{\xi} \, \mathrm{d}\nu(A) \, \mathrm{d}\mathbf{y}.$$

Moving the $\mathrm{d}\mathbf{y}$-integral inside yields a Dirac at $A^{-1}\boldsymbol{\xi} = \boldsymbol{\kappa}$. Hence,

$$E_{3D}(\boldsymbol{\kappa}) = \frac{1}{2} \int_{O(3)} |\hat{\mathbf{u}}(A\boldsymbol{\kappa})|^2 \, \mathrm{d}\nu(A).$$

It is a classical result that the push-forward of the Haar measure $\mathrm{d}\nu$ through the map $T_{\boldsymbol{\kappa}} : A \to A\boldsymbol{\kappa}$ is the normalized surface measure on the sphere $\kappa\mathbb{S}^2$, $\kappa = |\boldsymbol{\kappa}|$. We thus obtain

$$E_{3D}(\boldsymbol{\kappa}) = \frac{1}{8\pi\kappa^2} \int_{\kappa\mathbb{S}^2} |\hat{\mathbf{u}}(\boldsymbol{\xi})|^2 \, \mathrm{d}\boldsymbol{\xi}.$$

This leads to the classical energy spectrum definition

$$E(\kappa) = 4\pi\kappa^2 E_{3D}(\boldsymbol{\kappa}) = \frac{1}{2} \int_{\kappa\mathbb{S}^2} |\hat{\mathbf{u}}(\boldsymbol{\xi})|^2 \, \mathrm{d}\boldsymbol{\xi}.$$

Thus, the total energy is recovered from the spectrum by

$$\int_0^\infty E(\kappa) \, \mathrm{d}\kappa = \mathcal{E}.$$

We now relate $E(\kappa)$ and the second order structure function. Note that

$$\Gamma(\mathbf{y}) = \mathcal{E} + \frac{1}{2} \int_{O(3)} \langle (\mathbf{u}(\cdot + A\mathbf{y}) - \mathbf{u}(\cdot)) \cdot \mathbf{u}(\cdot) \rangle \, \mathrm{d}\nu(A).$$

Shifting inside the average by $A\mathbf{y}$ and reversing the sign of $\mathbf{y}$ we obtain

$$\Gamma(\mathbf{y}) = \mathcal{E} - \frac{1}{4} \int_{O(3)} \langle |\mathbf{u}(\cdot + A\mathbf{y}) - \mathbf{u}(\cdot)|^2 \rangle \, \mathrm{d}\nu(A).$$

The obtained average is the second order structure function

$$S_2(\ell) = \frac{1}{4} \int_{O(3)} \langle |\mathbf{u}(\cdot + A\mathbf{y}) - \mathbf{u}(\cdot)|^2 \rangle \, \mathrm{d}\nu(A), \quad \ell = |\mathbf{y}|.$$

One can now relate scaling laws of $S_2(\ell)$ to those of the energy spectrum.

Proposition 4.1. *Suppose $S_2(\ell) = c\ell^\alpha$ for all $\ell < \ell_0$ and some $0 < \alpha < 2$. Then*

(84)
$$E(\kappa) = \frac{c(\kappa)}{\kappa^{1+\alpha}}, \quad \text{where } c(\kappa) \to c_0 > 0 \text{ as } \kappa \to \infty.$$

Conversely, if $E(\kappa) = c\kappa^{-1-\alpha}$ for all $\kappa \geqslant \kappa_0$, then

(85)
$$S_2(\ell) = c(\ell)\ell^\alpha, \quad \text{where } c(\ell) \to c_0 > 0 \text{ as } \ell \to 0.$$

We note that the result actually holds for all $\alpha > 0$ with respectively stronger assumptions on the decay of the correlation function. We restrict ourselves to $\alpha < 2$ since in the extreme intermittency event we only have $\alpha = \frac{5}{3}$. For the Kolmogorov regime we have $S_2(\ell) = (\varepsilon\ell)^{2/3}$, hence $E(\kappa) \sim \kappa^{-5/3}$.

Proof. Using that Γ is a radial function, one can compute the Fourier integral of Γ as a function of scalar ℓ as follows

$$E(\kappa) = c \int_0^\infty \kappa\ell \sin(\kappa\ell) \Gamma(\ell) \, \mathrm{d}\ell.$$

Let us assume w.l.o.g. that $\ell_0 = 1$. Let $0 \leqslant \chi \leqslant 1$ be a smooth cutoff function with $\operatorname{supp}\chi \subset \{0 < \ell < 1\}$ and $\chi(\ell) = 1$ for $\ell \leqslant \frac{1}{2}$. We can write

$$E(\kappa) = c \int_0^\infty \kappa\ell \sin(\kappa\ell)\Gamma(\ell)\chi(\ell) \, d\ell + R(\kappa),$$

where

$$R(\kappa) = c \int_0^\infty \kappa\ell \sin(\kappa\ell)\Gamma(\ell)[1 - \chi(\ell)] \, d\ell.$$

Using that $\frac{1}{\kappa^4} \frac{d^4}{d\ell^4} \sin(k\ell) = \sin(k\ell)$, and integrating by parts we obtain

$$R(\kappa) = \frac{c}{\kappa^3} \int_0^\infty \sin(\kappa\ell)[\ell\Gamma(\ell)(1 - \chi(\ell))]^{(4)} \, d\ell.$$

Note that the under our assumption (82), the integral converges absolutely, and thus $|R(\kappa)| \lesssim \frac{c}{\kappa^3}$, which is of smaller order. Continuing with the main term we first split

$$\int_0^1 \kappa\ell \sin(\kappa\ell)\Gamma(\ell)\chi(\ell) \, d\ell = \mathcal{E} \int_0^1 \kappa\ell \sin(\kappa\ell)\chi(\ell) \, d\ell - \kappa \int_0^1 \sin(\kappa\ell)\ell^{1+\alpha}\chi(\ell) \, d\ell.$$

The first integral decays as an arbitrary power of κ which can be seen by performing similar computation as above. For the second we integrate by parts twice:

$$(86) \qquad -\kappa \int_0^1 \sin(\kappa\ell)\ell^{1+\alpha}\chi(\ell) \, d\ell = -\int_0^1 \cos(\kappa\ell)(\ell^{1+\alpha}\chi(\ell))' \, d\ell = \frac{1}{\kappa} \int_0^1 \sin(\kappa\ell)(\ell^{1+\alpha}\chi(\ell))'' \, d\ell.$$

Let us consider the case $0 < \alpha < 1$ at this point. Expanding

$$(\ell^{1+\alpha}\chi(\ell))'' = (1 + \alpha)\alpha\ell^{-1+\alpha}\chi(\ell) + 2(1 + \alpha)\ell^\alpha\chi'(\ell) + \ell^{1+\alpha}\chi''(\ell)$$

we can see that the latter two terms are supported away from the origin. We can apply the same integration by parts as before to show that those two terms decay as any power of κ. Finally,

$$\frac{1}{\kappa} \int_0^1 \sin(\kappa\ell)\ell^{-1+\alpha}\chi(\ell) \, d\ell = \frac{1}{\kappa^{1+\alpha}} \int_0^\kappa \sin(x)x^{-1+\alpha}\chi(\ell/\kappa) \, d\ell.$$

Denoting $c(\kappa) = \int_0^\kappa \sin(x)x^{-1+\alpha}\chi(\ell/\kappa) \, dx$, due to condition $0 < \alpha < 1$ we can see that the integral results in a convergent alternating series with vanishing error term. Moreover, $\lim_{\kappa \to \infty} c(\kappa) = \int_0^\infty \sin(x)x^{-1+\alpha} \, dx > 0$. This finishes the case $0 < \alpha < 1$.

In the case $\alpha = 1$, we integrate by parts again in (86):

$$\frac{1}{\kappa} \int_0^1 \sin(\kappa\ell)(\ell^2\chi(\ell))'' \, d\ell = \frac{1}{k^2} + \frac{1}{k^2} \int_0^1 \cos(\kappa\ell)(\ell^2\chi(\ell))''' \, d\ell.$$

Now the integrand is supported away from the origin and hence decays at least as $1/k^3$.

In the case $1 < \alpha < 2$ we obtain

$$\frac{1}{\kappa} \int_0^1 \sin(\kappa\ell)(\ell^{1+\alpha}\chi(\ell))'' \, d\ell = \frac{1}{k^2} \int_0^1 \cos(\kappa\ell)(\ell^{1+\alpha}\chi(\ell))''' \, d\ell.$$

Here the main term is

$$\int_0^1 \cos(\kappa\ell)\ell^{-2+\alpha}\chi(\ell) \, d\ell \sim \frac{1}{\kappa^{1+\alpha}}.$$

This proves the first part.

The converse statement follows similarly by taking inverse Fourier transform of (83).

□

References

[AGHA84]   F. Anselmet, Y. Gagne, E. J. Hopfinger, and R. A. Antonia. High-order velocity structure functions in turbulent shear flows. *Journal of Fluid Mechanics*, 140:63–89, 1984.

[BadVGG18] Hugo Beirão da Veiga, Yoshikazu Giga, and Zoran Grujić. Vorticity direction and regularity of solutions to the Navier-Stokes equations. In *Handbook of mathematical analysis in mechanics of viscous fluids*, pages 901–932. Springer, Cham, 2018.

[BFG19]   Zachary Bradshaw, Aseel Farhat, and Zoran Grujić. An algebraic reduction of the 'scaling gap' in the Navier-Stokes regularity problem. *Arch. Ration. Mech. Anal.*, 231(3):1983–2005, 2019.

[CD19]   Alexey Cheskidov and Mimi Dai. Kolmogorov's dissipation number and the number of degrees of freedom for the 3D Navier–Stokes equations. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 149(2):429–446, 2019.

[CD21]   A. Cheskidov and M. Dai. The number of degrees of freedom for the 2D Navier-Stokes equation: a connection with Kraichnan's theory of turbulence. *arXiv:math/2112.11606*, 2021.

[CS14a]   A. Cheskidov and R. Shvydkoy. Euler equations and turbulence: analytical approach to intermittency. *SIAM J. Math. Anal.*, 46(1):353–374, 2014.

[CS14b]   A. Cheskidov and R. Shvydkoy. A unified approach to regularity problems for the 3D Navier-Stokes and Euler equations: the use of Kolmogorov's dissipation range. *J. Math. Fluid Mech.*, 16(2):263–273, 2014.

[DF02]   Charles R. Doering and Ciprian Foias. Energy dissipation in body-forced turbulence. *Journal of Fluid Mechanics*, 467:289–306, 2002.

[DR00]   J. Duchon and R. Robert. Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations. *Nonlinearity*, 13(1):249–255, 2000.

[Dri21]   Theodore Drivas. Self-regularization in turbulence from the Kolmogorov 4/5-law and alignment. *arXiv:math/2111.03493*, 2021.

[ES06]   G. L. Eyink and K. R. Sreenivasan. Onsager and the theory of hydrodynamic turbulence. *Rev. Modern Phys.*, 78(1):87–135, 2006.

[FGC+21]   H. Faller, D. Geneste, T. Chaabo, A. Cheminet, V. Valori, Y. Ostovan, L. Cappanera, Ch. Cuvier, F. Daviaud, J.-M. Foucaut, and et al. On the nature of intermittency in a turbulent von Kármán flow. *Journal of Fluid Mechanics*, 914:A2, 2021.

[Fri95]   U. Frisch. *Turbulence*. Cambridge University Press, Cambridge, 1995. The legacy of A. N. Kolmogorov.

[FSN78]   U. Frisch, P.-L. Sulem, and M. Nelkin. A simple dynamical model of intermittent fully developed turbulence. *J. Fluid Mech.*, 87:719–736, 1978.

[Gra14]   Loukas Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.

[ISY20]   Kartik P. Iyer, Katepalli R. Sreenivasan, and P. K. Yeung. Scaling exponents saturate in three-dimensional isotropic turbulence. *Phys. Rev. Fluids*, 5:054605, May 2020.

[Jaf97]   S. Jaffard. Multifractal formalism for functions. II. Self-similar functions. *SIAM J. Math. Anal.*, 28(4):971–998, 1997.

[Jaf00]   Stéphane Jaffard. On the Frisch-Parisi conjecture. *J. Math. Pures Appl. (9)*, 79(6):525–552, 2000.

[Kol91]   A. N. Kolmogorov. The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. *Proc. Roy. Soc. London Ser. A*, 434(1890):9–13, 1991. Translated from the Russian by V. Levin, Turbulence and stochastic processes: Kolmogorov's ideas 50 years on.

[LBAC04]   Bruno Lashermes, Christophe Baudet, Patrice Abry, and Pierre Chainais. Limitation of scaling exponent estimation in turbulence. 2004.

[NLK+19]   F. Nguyen, J.-P. Laval, P. Kestener, A. Cheskidov, R. Shvydkoy, and B. Dubrulle. Local estimates of hölder exponents in turbulent vector fields. *Phys. Rev. E*, 99:053114, May 2019.

[PF83]   G. Parisi and U. Frisch. On the singularity structure of fully developed turbulence. In G. Parisi M. Ghil, R. Benzi, editor, *Turbulence and Predictability in Geophysical Fluid Dynamics. Proceed. Intern.School of Phys. 'E. Fermi'*, pages 84–87. North-Holland, Amsterdam, Varenna, Italy, 1983.

[SKF+16]   E. W. Saw, D. Kuzzay, D. Faranda, A. Guittonneau, F. Daviaud, C. Wiertel-Gasquet, V. Padilla, and B. Dubrulle. Experimental characterization of extreme events of inertial dissipation in a turbulent swirling flow. *Nature Communications*, 7(1):12466, 2016.

[SS93]   G. Stolovitzky and K. R. Sreenivasan. Scaling of structure functions. *Phys. Rev. E*, 48:R33–R36, Jul 1993.

[SVB+96]   K. R. Sreenivasan, S. I. Vainshtein, R. Bhiladvala, I. San Gil, S. Chen, and N. Cao. Asymmetry of velocity increments in fully developed turbulence and the scaling of low-order moments. *Phys. Rev. Lett.*, 77:1488–1491, Aug 1996.

[Vil03]   Cédric Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, and School of Mathematics, Institute for Advanced Study, Princeton.

Email address: `acheskid@uic.edu`

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, and Isaac Newton Institute for Mathematical Sciences, Cambridge.

Email address: `shvydkoy@uic.edu`