UNIFORM GLOBAL ATTRACTORS FOR THE NONAUTONOMOUS 3D NAVIER-STOKES EQUATIONS

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ABSTRACT. We obtain the existence and the structure of the weak uniform (with respect to the initial time) global attractor and construct a trajectory attractor for the 3D Navier-Stokes equations (NSE) with a fixed time-dependent force satisfying a translation boundedness condition. Moreover, we show that if the force is normal and every complete bounded solution is strongly continuous, then the uniform global attractor is strong, strongly compact, and solutions converge strongly toward the trajectory attractor. Our method is based on taking a closure of the autonomous evolutionary system without uniqueness, whose trajectories are solutions to the nonautonomous 3D NSE. The established framework is general and can also be applied to other nonautonomous dissipative partial differential equations for which the uniqueness of solutions might not hold. It is not known whether previous frameworks can also be applied in such cases as we indicate in open problems related to the question of uniqueness of the Leray-Hopf weak solutions.

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1. INTRODUCTION

The theory of uniform attractors of nonautonomous infinite-dimensional dissipative dynamical system bears its roots in the work of Haraux [Ha91], who defined the uniform global attractor as a minimal closed set which attracts all the trajectories starting from a bounded set uniformly with respect to (w.r.t.) the initial time. This naturally generalizes the notion of a global attractor to nonautonomous dynamical systems. In this paper we will present a method for obtaining the structure of the uniform global attractor of a general nonautonomous system. In particular, we obtain the existence and the structure of the weak uniform global attractor and construct a trajectory attractor for the 3D Navier-Stokes equations (NSE) with a fixed time-dependent force satisfying a translation boundedness condition.

Previous studies of uniform global attractors mostly used tools developed by Chepyzhov and Vishik [CV94, CV02]. Their framework was based on the use

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of the so-called time symbol (e.g. the external force in 3D NSE) and constructing a symbol space as a suitable closure of the translation family of the original symbol. To describe the structure of the uniform global attractor they introduced an auxiliary notion of a uniform w.r.t. the symbol space attractor. However, this method requires a strong condition on the force and only provides the structure of the uniform w.r.t. the symbol space attractor, which does not always have to coincide with the original uniform global attractor (see Open Problem 6.5). In this paper we present a different approach that deals directly with the notion of a uniform global attractor and is based on taking a closure of the family of trajectories of the system, which does not change the uniform global attractor. The established method is general and can be applied to any nonautonomous dissipative PDE.

Since the pioneering work of Leray, the problem of regularity of the 3D NSE has been a subject of serious investigation and still poses an important challenge for mathematicians. Due to the lack of a uniqueness proof, it is not known whether the 3D NSE possesses, for the autonomous case, a semigroup of, or for the nonautonomous case, a process of solution operators. Therefore, a classical theory of semigroup or process [H88, T88, L91, SY02, CV02] cannot be used for this system. A mathematical object describing long time behavior of an autonomous 3D NSE is a (weak) global attractor, a notion that goes all the way back to the seminal work by Foias and Temam [FT85].

The goal of this paper is to study the long time behavior of the 3D NSE with a fixed time-dependent force in the physical space without making any assumptions on weak solutions. Moreover, we assume that the force only satisfies a translation boundedness condition, which is the weakest conditions that guaranties the existence of a bounded uniform absorbing ball. In order to obtain the the structure of the weak uniform global attractor we will consider an autonomous evolutionary system without uniqueness, whose trajectories are solutions to the nonautonomous 3D NSE. The evolutionary system $\mathcal{E}$ was first introduced in [CF06] to study a weak global attractor and a trajectory attractor for the autonomous 3D NSE, and then the theory was developed further in [C09] to make it applicable to arbitrary autonomous dissipative PDE without uniqueness. In particular, it was shown that the global attractor consists of points on complete bounded trajectories under an assumption (see $\overline{\text{A}}$1) satisfied by autonomous PDEs. The evolutionary system is close to Ball's generalized semiflow [B98], but due to relaxed assumptions on the trajectories, the Leray-Hopf weak solutions of the 3D NSE always form an evolutionary system regardless whether they lose regularity or not. The advantage of this framework lies in a simultaneous use of weak and strong metric, which makes it applicable to any other PDE for which the uniqueness of solutions may be in limbo.

In [CL09] the authors already generalized the framework of the evolutionary system to study the long time behavior of nonautonomous dynamical systems without uniqueness. In this paper we develop the theory further and introduce a "closure of the evolutionary system" in order to obtain the structure of the uniform global attractor. This method avoids the necessity of constructing a symbol space and works for systems without uniqueness.

The paper is organized as follows. In Section 2 we briefly recall the theory of evolutionary system originally designed for autonomous systems, define a nonautonomous evolutionary system, and reduce it to an autonomous system. Then we consider classical cases of a process and a family of processes and show how they define evolutionary systems. In particular, when the evolutionary system is defined by a process, the uniform global attractor for the evolutionary system is identical to the uniform global attractor for the process. Hence, using the theory developed in this paper, we can describe the structure of the uniform global attractor of a general process, which was mentioned as an open problem in [CV94, CV02].

Section 3 is mainly concerned with the existence and the structure of the weak and strong uniform global attractors for the nonautonomous evolutionary system corresponding to the original equations under consideration, for instance, the 3D NSE with a fixed time-dependent external force. To this end, we consider a closure of the evolutionary system and prove that its weak uniform global attractor is identical to the one for the original evolutionary system. We then apply the theory developed in [C09] to the closure of the evolutionary system to obtain various properties of the uniform global attractor for the original system.

In Subsection 3.1 we use our framework to examine the notion of uniform w.r.t. symbol space global attractor. For this we assume that a suitable symbol space $\bar{\Sigma}$ is provided, and consider a nonautonomous evolutionary system with such symbol space satisfying the uniqueness condition, i.e., we assume that for a fixed symbol there exists only one trajectory starting at a given point. We then study the relation between the system and its evolutionary subsystem whose symbol space Σ is a dense subset of the former symbol space $\bar{\Sigma}$. Assuming that the system satisfies an additional condition (which translates into the strong translation compactness condition on the force in the case of the 3D NSE), we show that it has the same weak uniform global attractor as its subsystem. Therefore, we show that the uniform w.r.t. $\bar{\Sigma}$ attractor coincides with the uniform global attractor. Furthermore, if the evolutionary system is asymptotically compact, then the weak uniform global attractor is in fact the strong uniform global attractor. The results in this subsection generalize those in [CV02, LWZ05, Lu06, Lu07].

In Section 4 we study the notion of a trajectory attractor that was first introduced in [Se96] and further studied in [CV97, CV02, SY02]. We use the tools in the preceding sections to show the existence and the structure of the trajectory attractor of a nonatonomous evolutionary system such as the 3D NSE. Note that such a result could not be obtained with previous frameworks (see Open Problem 6.7). We also show a relation of the trajectory attractor to a uniform global attractor. Moreover, in the case where all complete bounded trajectories are strongly continuous, a strong convergence of the trajectories to the trajectory attractor is proved.

In Section 5 we consider the 3D NSE with a translation bounded in $L^2_{\rm loc}(\mathbb{R}; V')$ fixed time-dependent external force g_0. We show that the Leray-Hopf weak solutions form an evolutionary system investigated in Sections 3, and therefore all the results obtained in that section hold for the 3D NSE. In particular, we obtain the structure of the weak uniform global attractor $\mathcal{A}$. Notice again that we only require the weakest boundedness condition on the force and make no assumptions on the solutions of the 3D NSE. In addition, we show that if the force g_0 is normal and every complete bounded solution is strongly continuous, then the weak uniform global attractor is strong, strongly compact, and solutions converge strongly toward the trajectory attractor. The normality condition on the external force, introduced in [LWZ05] and Lu [Lu06], is weaker than the usual strong translation compactness condition (see [LWZ05]), which is generally required in applications of Chepyzhov and Vishik's approach [CV94, CV02].

Evolutionary systems are constructed without a suitable symbol space in Section 5. In Section 6 we construct a uniform w.r.t. symbol space global attractor for the 3D NSE. For this we first have to impose a stronger condition on the external force, namely, we assume that the force is strongly translation compact. Then we can find a suitable closure of the symbol space $\bar{\Sigma}$ for which the corresponding evolutionary system enjoys the desired compactness property, and hence obtain the structure of the uniform w.r.t. symbol space $\bar{\Sigma}$ attractor $\mathcal{A}_{\rm w}^{\bar{\Sigma}}$. However, this attractor might not coincide with the uniform global attractor $\mathcal{A}$ if Leray-Hopf weak solutions are not unique (see the Open Problem 6.5), which illustrates a limitation of the framework of uniform w.r.t. symbol global attractor put forward in [CV94] (see also [CV02]). It is still not clear how to obtain the structure of the uniform global attractor $\mathcal{A}$ using the notion of uniform w.r.t. symbol global attractor $\mathcal{A}_{\rm w}^{\bar{\Sigma}}$ for the 3D NSE or other systems without uniqueness, or where the uniqueness is not known. Similarly, the trajectory attractors constructed in [Se96] and [CV97, CV02] for the 3D NSE or other systems without uniqueness, might be bigger than the one we constructed in the section (see Open Problem 6.7).

2. EVOLUTIONARY SYSTEM

2.1. **Autonomous case.** Here we recall the basic definitions and results on evolutionary systems (see [C09] for details). Let $(X, \mathrm{d_s}(\cdot,\cdot))$ be a metric space endowed with a metric $\mathrm{d_s}$, which will be referred to as a strong metric. Let $\mathrm{d_w}(\cdot,\cdot)$ be another metric on X satisfying the following conditions:

(1) X is $\mathrm{d_w}$-compact.
(2) If $\mathrm{d_s}(u_n, v_n) \to 0$ as $n \to \infty$ for some $u_n, v_n \in X$, then $\mathrm{d_w}(u_n, v_n) \to 0$ as $n \to \infty$.

Due to the property 2, $\mathrm{d_w}(\cdot,\cdot)$ will be referred to as a weak metric on X. Denote by $\overline{A}^{\bullet}$ the closure of a set $A \subset X$ in the topology generated by $\mathrm{d}_\bullet$. Note that any strongly compact ($\mathrm{d_s}$-compact) set is weakly compact ($\mathrm{d_w}$-compact), and any weakly closed set is strongly closed.

Let $C([a,b];X_\bullet)$, where $\bullet = \mathrm{s}$ or w, be the space of $\mathrm{d}_\bullet$-continuous X-valued functions on $[a,b]$ endowed with the metric

$$\mathrm{d}_{C([a,b];X_\bullet)}(u,v) := \sup_{t\in[a,b]} \mathrm{d}_\bullet(u(t),v(t)).$$

Let also $C([a,\infty);X_\bullet)$ be the space of $\mathrm{d}_\bullet$-continuous X-valued functions on $[a,\infty)$ endowed with the metric

$$\mathrm{d}_{C([a,\infty);X_\bullet)}(u,v) := \sum_{T\in\mathbb{N}} \frac{1}{2^T} \frac{\sup\{\mathrm{d}_\bullet(u(t),v(t)) : a \le t \le a+T\}}{1+\sup\{\mathrm{d}_\bullet(u(t),v(t)) : a \le t \le a+T\}}.$$

Let

$$\mathcal{T} := \{I : \ I = [T,\infty) \subset \mathbb{R}, \text{ or } I = (-\infty,\infty)\},$$

and for each $I \subset \mathcal{T}$, let $\mathcal{F}(I)$ denote the set of all X-valued functions on I. Now we define an evolutionary system $\mathcal{E}$ as follows.

Definition 2.1. *A map $\mathcal{E}$ that associates to each $I \in \mathcal{T}$ a subset $\mathcal{E}(I) \subset \mathcal{F}$ will be called an evolutionary system if the following conditions are satisfied:*

(1) $\mathcal{E}([0,\infty)) \neq \emptyset$.
(2) $\mathcal{E}(I+s) = \{u(\cdot) : \ u(\cdot - s) \in \mathcal{E}(I)\}$ *for all* $s \in \mathbb{R}$.
(3) $\{u(\cdot)|_{I_2} : u(\cdot) \in \mathcal{E}(I_1)\} \subset \mathcal{E}(I_2)$ *for all pairs* $I_1, I_2 \in \mathcal{T}$*, such that* $I_2 \subset I_1$.
(4) $\mathcal{E}((-\infty,\infty)) = \{u(\cdot) : \ u(\cdot)|_{[T,\infty)} \in \mathcal{E}([T,\infty)) \ \forall T \in \mathbb{R}\}$.

We will refer to $\mathcal{E}(I)$ as the set of all trajectories on the time interval I. Trajectories in $\mathcal{E}((-\infty,\infty))$ are called complete. Let $P(X)$ be the set of all subsets of X. For every $t \ge 0$, define a map

$$R(t) : P(X) \to P(X),$$
$$R(t)A := \{u(t) : u(0) \in A, u \in \mathcal{E}([0,\infty))\}, \qquad A \subset X.$$

Note that the assumptions on $\mathcal{E}$ imply that $R(t)$ enjoys the following property:

$$R(t+s)A \subset R(t)R(s)A, \qquad A \subset X, \quad t,s \ge 0. \tag{1}$$

Definition 2.2. *A set $\mathcal{A}_\bullet \subset X$ is a $\mathrm{d}_\bullet$-global attractor ($\bullet = \mathrm{s}, \mathrm{w}$) of if $\mathcal{A}_\bullet$ is a minimal set which is*

(1) $\mathrm{d}_\bullet$*-closed.*
(2) $\mathrm{d}_\bullet$*-attracting: for any $B \subset X$ and $\epsilon > 0$, there exists t_0, such that*

$$R(t)B \subset B_\bullet(\mathcal{A}_\bullet, \epsilon) := \{u : \inf_{x\in\mathcal{A}_\bullet} \mathrm{d}_\bullet(u,x) < \epsilon\}, \quad \forall\, t \ge t_0.$$

Definition 2.3. *The* $\omega_\bullet$*-limit (*$\bullet = \mathrm{s}, \mathrm{w}$*) of a set* $A \subset X$ *is*

$$\omega_\bullet(A) := \bigcap_{T\geq 0} \overline{\bigcup_{t\geq T} R(t)A}^{\bullet}.$$

An equivalent definition of the $\omega_\bullet$-limit set is given by

$$\omega_\bullet(A) = \{x \in X : \text{ there exist sequences } t_n \to \infty \text{ as } n \to \infty \text{ and } x_n \in R(t_n)A, \\ \text{such that } x_n \to x \text{ in } \mathrm{d}_\bullet\text{-metric as } n \to \infty\}.$$

In order to extend the notion of invariance from a semiflow to an evolutionary system we use the following mapping:

$$\widetilde{R}(t)A := \{u(t) : u(0) \in A, u \in \mathcal{E}((-\infty, \infty))\}, \qquad A \subset X,\ t \in \mathbb{R}.$$

Definition 2.4. *A set* $A \subset X$ *is positively invariant if*

$$\widetilde{R}(t)A \subset A, \qquad \forall t \geq 0.$$

A *is invariant if*

$$\widetilde{R}(t)A = A, \qquad \forall t \geq 0.$$

A *is quasi-invariant if for every* $a \in A$ *there exists a complete trajectory* $u \in \mathcal{E}((-\infty, \infty))$ *with* $u(0) = a$ *and* $u(t) \in A$ *for all* $t \in \mathbb{R}$*.*

Definition 2.5. *The evolutionary system* $\mathcal{E}$ *is asymptotically compact if for any* $t_k \to \infty$ *and any* $x_k \in R(t_k)X$*, the sequence* $\{x_k\}$ *is relatively strongly compact.*

Below are some additional assumptions that we will impose on $\mathcal{E}$ in some cases.

- $\bar{\text{A}}$1 $\mathcal{E}([0, \infty))$ is a compact set in $C([0, \infty); X_{\mathrm{w}})$.
- $\bar{\text{A}}$2 (Energy inequality) Assume that X is a set in some Banach space H satisfying the Radon-Riesz property with the norm denoted by $|\cdot|$, such that $\mathrm{d_s}(x, y) = |x - y|$ for $x, y \in X$ and $\mathrm{d_w}$ induces the weak topology on X. Assume also that for any $\epsilon > 0$, there exists δ, such that for every $u \in \mathcal{E}([0, \infty))$ and $t > 0$,
$$|u(t)| \leq |u(t_0)| + \epsilon,$$
for t_0 a.e. in $(t - \delta, t)$.
- $\bar{\text{A}}$3 (Strong convergence a.e.) Let $u, u_n \in \mathcal{E}([0, \infty))$, be such that $u_n \to u$ in $C([0, T]; X_{\mathrm{w}})$ for some $T > 0$. Then $u_n(t) \to u(t)$ strongly a.e. in $[0, T]$.

Remark 2.6. *A Banach space* H *is said to satisfy the Radon-Riesz property when a sequence converges if and only if it converges weakly and the norms of the elements of the sequence converge to the norm of the weak limit. In many applications* X *is a bounded closed set in a uniformly convex separable Banach space* H*. Then the weak topology of* H *is metrizable on* X*, and* X *is compact with respect to such a metric* $\mathrm{d_w}$*. Moreover, the Radon-Riesz property is automatically satisfied in this case.*

Theorem 2.7. [C09] *Let $\mathcal{E}$ be an evolutionary system. Then*

1. *The weak global attractor $\mathcal{A}_{\mathrm{w}}$ exists.*

Furthermore, if $\mathcal{E}$ satisfies $\bar{A}1$, then

2. $\mathcal{A}_{\mathrm{w}} = \omega_{\mathrm{w}}(X) = \omega_{\mathrm{s}}(X) = \{u_0 : u_0 = u(0)$ *for some* $u \in \mathcal{E}((-\infty, \infty))\}$.
3. $\mathcal{A}_{\mathrm{w}}$ *is the maximal invariant and maximal quasi-invariant set.*
4. *(Weak uniform tracking property) For any $\epsilon > 0$, there exists t_0, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0, \infty))$ satisfies $\mathrm{d}_{C([t^*,\infty);X_{\mathrm{w}})}(u, v) < \epsilon$, for some complete trajectory $v \in \mathcal{E}((-\infty, \infty))$.*

Theorem 2.8. [C09] *Let $\mathcal{E}$ be an asymptotically compact evolutionary system. Then*

1. *The strong global attractor $\mathcal{A}_{\mathrm{s}}$ exists, it is strongly compact, and $\mathcal{A}_{\mathrm{s}} = \mathcal{A}_{\mathrm{w}}$.*

Furthermore, if $\mathcal{E}$ satisfies $\bar{A}1$, then

2. *(Strong uniform tracking property) for any $\epsilon > 0$ and $T > 0$, there exists t_0, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0, \infty))$ satisfies $\mathrm{d}_{\mathrm{s}}(u(t), v(t)) < \epsilon$, $\forall t \in [t^*, t^* + T]$, for some complete trajectory $v \in \mathcal{E}((-\infty, \infty))$.*

Theorem 2.9. [C09] *Let $\mathcal{E}$ be an evolutionary system satisfying $\bar{A}1$, $\bar{A}2$, and $\bar{A}3$ and such that every complete trajectory is strongly continuous. Then $\mathcal{E}$ is asymptotically compact.*

2.2. **Nonautonomous case.** In this subsection we will show that the notion of evolutionary system is naturally applicable to a nonautonomous system.

Let Σ be a parameter set and $\{T(s)|s \geq 0\}$ be a family of operators acting on Σ satisfying $T(s)\Sigma = \Sigma$, $\forall s \geq 0$. Any element $\sigma \in \Sigma$ will be called (time) symbol and Σ will be called (time) symbol space. For instance, in many applications $\{T(s)\}$ is the translation semigroup and Σ is the translation family of the time dependent items of the considered system or its closure in some appropriate topological space.

Definition 2.10. *A family of maps $\mathcal{E}_\sigma$, $\sigma \in \Sigma$ that for very $\sigma \in \Sigma$ associates to each $I \in \mathcal{T}$ a subset $\mathcal{E}_\sigma(I) \subset \mathcal{F}(I)$ will be called a nonautonomous evolutionary system if the following conditions are satisfied:*

(1) $\mathcal{E}_\sigma([\tau, \infty)) \neq \emptyset, \forall \tau \in \mathbb{R}$.
(2) $\mathcal{E}_\sigma(I + s) = \{u(\cdot) : u(\cdot - s) \in \mathcal{E}_{T(s)\sigma}(I)\}, \forall s \geq 0$.
(3) $\{u(\cdot)|_{I_2} : u(\cdot) \in \mathcal{E}_\sigma(I_1)\} \subset \mathcal{E}_\sigma(I_2), \forall I_1, I_2 \in \mathcal{T}, I_2 \subset I_1$.
(4) $\mathcal{E}_\sigma((-\infty, \infty)) = \{u(\cdot) : u(\cdot)|_{[\tau,\infty)} \in \mathcal{E}_\sigma([\tau, \infty)), \forall \tau \in \mathbb{R}\}$.

We will refer to $\mathcal{E}_\sigma(I)$ as the set of all trajectories with respect to (w.r.t.) the symbol σ on the time interval I. Trajectories in $\mathcal{E}_\sigma((-\infty, \infty))$ will be called complete w.r.t. σ. For every $t \geq \tau$, $\tau \in \mathbb{R}$, $\sigma \in \Sigma$, define a map

$$R_\sigma(t,\tau) : P(X) \to P(X),$$
$$R_\sigma(t,\tau)A := \{u(t) : u(\tau) \in A, u \in \mathcal{E}_\sigma([\tau,\infty))\}, \ \ A \subset X.$$

Similarly, the assumptions on $\mathcal{E}_\sigma$, $\sigma \in \Sigma$ imply that $R_\sigma(t,\tau)$ enjoys the following property:

$$R_\sigma(t,\tau)A \subset R_\sigma(t,s)R_\sigma(s,\tau)A, \ \ A \subset X, \qquad \forall t \geq s \geq \tau, \tau \in \mathbb{R}. \tag{2}$$

Let us now show how a nonautonomous evolutionary systems can be defined in the classical case where the uniqueness of trajectories holds.

Let H be a phase space (a separable reflexive Banach space). Consider a process of a two-parameter family of single-valued operators $U_{\sigma_0}(t,\tau) : H \to H$, satisfying the following conditions:

$$\begin{aligned} U_{\sigma_0}(t,s) \circ U_{\sigma_0}(s,\tau) = U_{\sigma_0}(t,\tau), \qquad &\forall\, t \geq s \geq \tau,\ \tau \in \mathbb{R}, \\ U_{\sigma_0}(\tau,\tau) = \text{Identity operator}, \qquad &\tau \in \mathbb{R}. \end{aligned} \tag{3}$$

Here σ_0 is a fixed symbol, which is usually the collection of all time-dependent terms of a considered system. So we assume that it is a function on $\mathbb{R}$ with values in some space. A trajectory u on $[\tau,\infty)$ is a mapping from $[\tau,\infty)$ to H, such that

$$u(t) = U_{\sigma_0}(t,\tau)u(\tau), \qquad t \geq \tau. \tag{4}$$

A ball $B \subset H$ is called a uniform (w.r.t. $\tau \in \mathbb{R}$) absorbing ball if for any bounded set $A \subset H$, there exists a $t_0 = t_0(A)$, such that,

$$\bigcup_{\tau\in\mathbb{R}} \bigcup_{t\geq t_0} U_{\sigma_0}(t+\tau,\tau)A \subset B. \tag{5}$$

Assume that the process is dissipative, i.e., there exists a uniformly (w.r.t. $\tau \in \mathbb{R}$) absorbing ball B. Since we are interested in a long-time behavior of solutions, it is enough to consider a restriction of the process to B. A uniform (w.r.t. $\tau \in \mathbb{R}$) attractor $\mathcal{A}$ of the process is a minimal closed set satisfying that, for any $A \subset B$ and $\epsilon > 0$ there exists $t_0 = t_0(\epsilon, A)$, such that

$$\bigcup_{\tau\in\mathbb{R}} \bigcup_{t\geq t_0} U_{\sigma_0}(t+\tau,\tau)A \subset B_H(\mathcal{A},\epsilon). \tag{6}$$

Now denote by Σ the translation family $\{\sigma_0(\cdot+h) | h \in \mathbb{R}\}$ of σ_0 and define

$$U_{\sigma_0(\cdot+h)}(t,\tau) := U_{\sigma_0(\cdot)}(t+h,\tau+h), \qquad \forall\, t \geq \tau,\ \tau \in \mathbb{R},\ h \in \mathbb{R}. \tag{7}$$

Due to the uniqueness of the trajectories, for any $\sigma \in \Sigma$, the family of operators $U_\sigma(t,\tau)$ define a process, i.e., (3) is valid with σ substituted for σ_0. Obviously, $T(s)\Sigma = \Sigma$, $\forall s \geq 0$ and the following translation identity holds,

$$U_\sigma(t+s,\tau+s) = U_{T(s)\sigma}(t,\tau), \qquad \forall\, \sigma \in \Sigma,\ t \geq \tau,\ \tau \in \mathbb{R},\ s \geq 0, \tag{8}$$

where $\{T(s)\}_{s\geq 0}$ is the translation semigroup. We now consider the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$. Note that (5) is equivalent to that for any $\tau \in \mathbb{R}$ and bounded set $A \subset H$ there exists $t_0 = t_0(A) \geq \tau$, such that

$$\bigcup_{\sigma\in\Sigma} \bigcup_{t\geq t_0} U_\sigma(t+\tau,\tau)A \subset B, \tag{9}$$

i.e., $B \subset H$ is also a uniform (w.r.t. $\sigma \in \Sigma$) absorbing ball for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$. Similarly, (6) is equivalent to that for any $\tau \in \mathbb{R}$, $A \subset B$ and $\epsilon > 0$ there exists $t_0 = t_0(\epsilon, A)$, such that

$$\bigcup_{\sigma\in\Sigma} \bigcup_{t\geq t_0} U_{\sigma_0}(t+\tau,\tau)A \subset B_H(\mathcal{A},\epsilon), \tag{10}$$

i.e, $\mathcal{A}$ is also a uniform (w.r.t. $\sigma \in \Sigma$) attractor for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$. Now take $X = B$. Note that since H is a separable reflexive Banach space, both the strong and the weak topologies on X are metrizable. Define the maps $\mathcal{E}_\sigma$, $\sigma \in \Sigma$ in the following way:

$$\mathcal{E}_\sigma([\tau,\infty)) := \{u(\cdot) : u(t) = U_\sigma(t,\tau)u_\tau, u_\tau \in X, t \geq \tau\}.$$

Conditions 1–4 in the definition of the nonautonomous evolutionary system $\mathcal{E}_\sigma$, $\sigma \in \Sigma$ follow from the definition of the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$. In addition, by (4), we have

$$R_\sigma(t,\tau)A = U_\sigma(t,\tau)A, \quad \forall\, A \subset X,\, \sigma \in \Sigma,\, t \geq \tau,\, \tau \in \mathbb{R}.$$

Hence, the process $\{U_{\sigma_0}(t,\tau)\}$ always defines a nonautonomous evolutionary system with the symbol space being the translation family Σ of σ_0.

In the theory of Chepyzhov & Vishik [CV94, CV02], when studying the existence and other properties, such as the invariance of the uniform (w.r.t. $\tau \in \mathbb{R}$) attractor of the process $\{U_{\sigma_0}(t,\tau)\}$, one considers a family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \bar{\Sigma}$ with the symbol space $\bar{\Sigma}$ being the closure of Σ in some appropriate topology space. Accordingly, one consider a family of equations with symbols in the strongly compact closure of the translation family Σ of the original symbol σ_0 in a corresponding functional space. In general, suppose that a family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \bar{\Sigma}$ satisfies the following natural translation identity:

$$U_\sigma(t+s,\tau+s) = U_{T(s)\sigma}(t,\tau), \qquad \forall\, \sigma \in \bar{\Sigma},\ t \geq \tau,\ \tau \in \mathbb{R},\ s \geq 0,$$

and $T(s)\bar{\Sigma} = \bar{\Sigma}$, $\forall\, s \geq 0$. Proceeding in a similar manner with $\bar{\Sigma}$ replacing Σ, it is easy to check that the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \bar{\Sigma}$ also defines a nonautonomous evolutionary system with symbol space $\bar{\Sigma}$.

2.3. **Reducing a nonautonomous evolutionary system to an autonomous evolutionary system.** In this subsection we show that any nonautonomous evolutionary system can be viewed as an (autonomous) evolutionary system. We start with the following key lemma.

Lemma 2.11. *Let $\tau_0 \in \mathbb{R}$ be fixed. Then for any $\tau \in \mathbb{R}$ and $\sigma \in \Sigma$, there exists at least one $\sigma' \in \Sigma$ such that*

$$\mathcal{E}_\sigma([\tau,\infty)) = \{u(\cdot) : u(\cdot - \tau + \tau_0) \in \mathcal{E}_{\sigma'}([\tau_0,\infty))\}. \tag{11}$$

Proof. i). Case $\tau \geq \tau_0$. Thanks to condition 2 in the definition of the nonautonomous evolutionary system we can just take $\sigma' = T(\tau - \tau_0)\sigma$.

ii). Case $\tau < \tau_0$. Since Σ is invariant, there exists at least one σ' such that $T(\tau_0 - \tau)\sigma' = \sigma$. Again, by condition 2 in the definition of $\mathcal{E}_\sigma$, $\sigma \in \Sigma$, we have

$$\mathcal{E}_{\sigma'}([\tau_0,\infty)) = \{u(\cdot) : u(\cdot - \tau_0 + \tau) \in \mathcal{E}_\sigma([\tau,\infty))\},$$

which is equivalent to (11). □

Remark 2.12. *In many applications, the elements of the symbol space Σ are functions on the real line and $\{T(s)\}_{s\geq 0}$ is the translation semigroup. If the existence of σ' is unique in Lemma 2.11, corresponding to the backward uniqueness property of the system, $\{T(s)\}_{s\geq 0}$ can be extended to a group and condition 2 in the definition of the nonautonomous evolutionary system is valid for $s \in \mathbb{R}$.*

It follows from Lemma 2.11 that

$$\bigcup_{\sigma\in\Sigma} R_\sigma(t,0)A = \bigcup_{\sigma\in\Sigma} R_\sigma(t+\tau,\tau)A, \quad \forall\, A \subset X, \tau \in \mathbb{R}, t \geq 0.$$

So it is convenient to denote

$$R_\Sigma(t)A := \bigcup_{\sigma\in\Sigma} R_\sigma(t,0)A, \quad \forall\, A \subset X,\ t \geq 0.$$

Similarly, we denote

$$\mathcal{E}_\Sigma(I) := \bigcup_{\sigma\in\Sigma} \mathcal{E}_\sigma(I), \quad \forall\, I \in \mathcal{T}.$$

Now we define an (autonomous) evolutionary system $\mathcal{E}$ in the following way:

$$\mathcal{E}(I) := \mathcal{E}_\Sigma(I), \quad \forall\, I \in \mathcal{T}.$$

It is easy to check that all the conditions in Definition 2.1 are satisfied. Moreover, for this evolutionary system we obviously have

$$R(t)A = R_\Sigma(t)A, \quad \forall\, A \subset X, t \geq 0. \tag{12}$$

Now the notions of invariance, quasi-invariance, and a global attractor for $\mathcal{E}$ can be extended to the nonautonomous evolutionary system $\{\mathcal{E}_\sigma\}_{\sigma\in\Sigma}$. For instance, the global attractors for evolutionary systems defined by a process and a family of processes in Section 2.2 are the uniform (w.r.t. the initial time) attractor and uniform (w.r.t. the symbol space) attractor, respectively. The global attractor in the nonautonomous case will be conventionally called a uniform global attractor (or simply a global attractor). Other than that we will not distinguish between autonomous and nonautonomous evolutionary systems and denote an evolutionary system with a symbol space Σ by $\mathcal{E}_\Sigma$ and its attractor by $\mathcal{A}^\Sigma$ if it is necessary.

The advantage of such an approach will be clear in next section, where we will see that for some evolutionary systems constructed from nonautonomous dynamical systems the associated symbol spaces are not known.

3. Closure of an evolutionary system

In this section we will investigate evolutionary systems $\mathcal{E}$ satisfying the following property:

A1 $\mathcal{E}([0,\infty))$ is a precompact set in $C([0,\infty);X_{\mathrm{w}})$.

In addition, we will present some results for evolutionary systems satisfying these additional properties:

A2 (Energy inequality) Assume that X is a set in some Banach space H satisfying the Radon-Riesz property with the norm denoted by $|\cdot|$, such that $\mathrm{d_s}(x,y) = |x-y|$ for $x,y \in X$ and $\mathrm{d_w}$ induces the weak topology on X. Assume also that for any $\epsilon > 0$, there exists δ, such that for every $u \in \mathcal{E}([0,\infty))$ and $t > 0$,

$$|u(t)| \leq |u(t_0)| + \epsilon,$$

for t_0 a.e. in $(t-\delta, t)$.

A3 (Strong convergence a.e.) Let $u_k \in \mathcal{E}([0,\infty))$, be such that u_k is $\mathrm{d}_{C([0,T];X_{\mathrm{w}})}$-Cauchy sequence in $C([0,T];X_{\mathrm{w}})$ for some $T > 0$. Then $u_k(t)$ is $\mathrm{d_s}$-Cauchy sequence a.e. in $[0,T]$.

Such kinds of evolutionary systems are closely related to the concept of the uniform w.r.t. the initial time global attractor for nonautonomous system, initiated by Haraux. For instance, as shown in previous section, the process $\{U_{\sigma_0}(t,\tau)\}$ defines an evolutionary system $\mathcal{E}_\Sigma$ whose uniform global attractor is the uniform w.r.t. the initial time global attractor due to Haraux. However, instead of condition $\bar{\text{A}}1$, $\mathcal{E}_\Sigma$ usually satisfies only A1. The Chepyzhov-Vishik approach requires finding a suitable closure $\bar{\Sigma}$ of the symbol space in some topological space. In [Lu07, CL09] open problems indicate that there may not exist a symbol space $\bar{\Sigma}$, such that a family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \bar{\Sigma}$ can be defined. Later we will see that even for evolutionary systems taking a closure of the symbol space is not always appropriate to study the uniform global attractor.

Denote by $\mathcal{A}_\bullet$ the uniform $\mathrm{d}_\bullet$-global attractor of $\mathcal{E}$. We will investigate the existence and the structure of $\mathcal{A}_\bullet$ using a new method that involves taking a closure of the evolutionary system $\mathcal{E}$. Let

$$\bar{\mathcal{E}}([\tau,\infty)) := \overline{\mathcal{E}([\tau,\infty))}^{C([\tau,\infty);X_{\mathrm{w}})}, \quad \forall\, \tau \in \mathbb{R}.$$

It can be checked that $\bar{\mathcal{E}}$ is also an evolutionary system. We call $\bar{\mathcal{E}}$ the closure of the evolutionary system $\mathcal{E}$, and add the top-script $^-$ to the corresponding notations in previous subsections for $\bar{\mathcal{E}}$. For instance, we denote by $\bar{\mathcal{A}}_\bullet$ the uniform $\mathrm{d}_\bullet$-global attractor for $\bar{\mathcal{E}}$.

First, we clearly have the following:

Lemma 3.1. *If $\mathcal{E}$ satisfies A1, then $\bar{\mathcal{E}}$ satisfies $\bar{\text{A1}}$.*

Now we will obtain the structure of the weak global attractor $\mathcal{A}_\mathrm{w}$ as well as the weak ω-limit of any weakly open set:

Theorem 3.2. *Let $\mathcal{E}$ be an evolutionary system satisfying A1. Then $\omega_\mathrm{w}(A) = \bar{\omega}_\mathrm{w}(A)$ for any weakly open set A in X. In particular,*

$$\mathcal{A}_\mathrm{w} = \bar{\mathcal{A}}_\mathrm{w} = \{u_0 : \ u_0 = u(0) \textit{ for some } u \in \bar{\mathcal{E}}((-\infty,\infty))\}.$$

Proof. From the definition of $\bar{\mathcal{E}}$ it follows that

$$\overline{\bigcup_{t\geq T} R(t)A}^{\mathrm{w}} \subset \overline{\bigcup_{t\geq T} \bar{R}(t)A}^{\mathrm{w}}, \quad \forall T \geq 0.$$

Hence, $\omega_\mathrm{w}(A) \subset \bar{\omega}_\mathrm{w}(A)$.

Now take any $x \in \bar{\omega}_\mathrm{w}(A)$. There exist sequences $t_n \to \infty$ as $n \to \infty$ and $x_n \in \bar{R}(t_n)A$, such that $x_n \to x$ in d_w-metric as $n \to \infty$. By definition of $\bar{\mathcal{E}}$ there exist $y_n \in R(t_n)A$ satisfying

$$\mathrm{d}_\mathrm{w}(y_n, x_n) \leq \frac{1}{n}.$$

Therefore,

$$\mathrm{d}_\mathrm{w}(y_n, x) \leq \mathrm{d}_\mathrm{w}(y_n, x_n) + \mathrm{d}_\mathrm{w}(x_n, x) \leq \frac{1}{n} + \mathrm{d}_\mathrm{w}(x_n, x) \to 0, \quad \text{as } n \to \infty, \tag{13}$$

which means that $x \in \omega_\mathrm{w}(A)$. Hence, $\bar{\omega}_\mathrm{w}(A) \subset \omega_\mathrm{w}(A)$. This concludes the first part of the proof.

The second part of the theorem follows from Theorem 2.7 and the facts that the weak global attractors $\mathcal{A}_\mathrm{w}$ and $\bar{\mathcal{A}}_\mathrm{w}$ exist and equal to $\omega_\mathrm{w}(X)$ and $\bar{\omega}_\mathrm{w}(X)$, respectively. □

If $\mathcal{E}$ is asymptotically compact then Theorem 2.8 immediately implies that the strong uniform global attractor $\mathcal{A}_\mathrm{s}$ exists and $\mathcal{A}_\mathrm{s} = \mathcal{A}_\mathrm{w}$. It also easy to see that the strong attracting property in Theorem 2.8 holds under the weaker assumption A1:

Theorem 3.3 (Strong uniform tracking property)**.** *Let $\mathcal{E}$ be an asymptotically compact evolutionary system satisfying A1. Let $\bar{\mathcal{E}}$ be the closure of the evolutionary system $\mathcal{E}$. Then for any $\epsilon > 0$ and $T > 0$, there exists t_0, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0,\infty))$ satisfies*

$$\mathrm{d}_\mathrm{s}(u(t), v(t)) < \epsilon, \quad \forall t \in [t^*, t^* + T],$$

for some complete trajectory $v \in \bar{\mathcal{E}}((-\infty,\infty))$.

Proof. Suppose to the contrary that there exist $\epsilon > 0$, $T > 0$, and sequences $u_n \in \mathcal{E}([0,\infty))$, $t_n \to \infty$ as $n \to \infty$, such that

$$\sup_{t\in[t_n,\,t_n+T]} \mathrm{d_s}(u_n(t), v(t)) \geq \epsilon, \quad \forall n, \tag{14}$$

for all $v \in \bar{\mathcal{E}}((-\infty,\infty))$.

On the other hand, since $\bar{\mathcal{E}}$ satisfies $\bar{\mathrm{A}}1$, the weak uniform tracking property in Theorem 2.7 implies that there exists a sequence $v_n \in \bar{\mathcal{E}}((-\infty,\infty))$, such that

$$\lim_{n\to\infty} \sup_{t\in[t_n,\,t_n+T]} \mathrm{d_w}(u_n(t), v_n(t)) = 0. \tag{15}$$

Thanks to (14), there exists a sequence $\hat{t}_n \in [t_n, t_n+T]$, such that

$$\mathrm{d_s}(u_n(\hat{t}_n), v_n(\hat{t}_n)) \geq \epsilon/2, \quad \forall n, \tag{16}$$

Now note that $\{u_n(\hat{t}_n)\}$ is relatively strongly compact due to the asymptotic compactness of $\mathcal{E}$. In addition, Theorem 3.2 implies that

$$\{v_n(\hat{t}_n)\} \subset \mathcal{A}_\mathrm{w}.$$

Thanks again to the asymptotic compactness of $\mathcal{E}$, $\mathcal{A}_\mathrm{w}$ is strongly compact due to Theorem 2.8. Hence, the sequence $\{v_n(\hat{t}_n)\}$ is also relatively strongly compact. Then it follows from (15) that the limits of the convergent subsequences of $\{u_n(\hat{t}_n)\}$ and $\{v_n(\hat{t}_n)\}$ coincide, which contradicts (16). □

Finally, in order to extend Theorem 2.9 to $\mathcal{E}$ we need the following:

Lemma 3.4. *If $\mathcal{E}$ satisfies A2 and A3, then $\bar{\mathcal{E}}$ satisfies $\bar{\mathrm{A}}2$ and $\bar{\mathrm{A}}3$.*

Proof. Clearly $\bar{\mathrm{A}}3$ holds by definition of $\bar{\mathcal{E}}$. Now take $u \in \bar{\mathcal{E}}([0,\infty))$ and $T > 0$. There exists a sequence $u_n \in \mathcal{E}([0,\infty))$ satisfying

$$u_n \to u \text{ in } C([0,T]; X_\mathrm{w}).$$

Thank to $\bar{\mathrm{A}}3$,

$$u_n(t) \to u(t) \text{ strongly in } [0,T] \setminus E_0,$$

where E_0 is a set of zero measure. Due to A2, for any $\epsilon > 0$, there exists δ, such that for every $u_n \in \mathcal{E}([0,\infty))$ and $T > 0$,

$$|u_n(T)| \leq |u_n(t)| + \epsilon,$$

for t in $(T-\delta, T) \setminus E_n$, where E_n is a zero measure set. Taking the lower limit as $n \to \infty$ we obtain

$$|u(T)| \leq \liminf |u_n(T)| \leq |u(t)| + \epsilon, \quad t \in (T-\delta, T) \setminus \cup_{i=0}^{\infty} E_i,$$

which means that $\bar{\mathrm{A}}2$ holds. □

With the above results in hand, we conclude with the following versions of Theorem 2.7, 2.8, and 2.9 for $\mathcal{E}$.

Theorem 3.5. *Let $\mathcal{E}$ be an evolutionary system. Then*

1. *The weak global attractor $\mathcal{A}_{\mathrm{w}}$ exists.*

Furthermore, assume that $\mathcal{E}$ satisfies A1. Let $\bar{\mathcal{E}}$ be the closure of $\mathcal{E}$. Then

2. $\mathcal{A}_{\mathrm{w}} = \omega_{\mathrm{w}}(X) = \bar{\omega}_{\mathrm{w}}(X) = \bar{\omega}_{\mathrm{s}}(X) = \bar{\mathcal{A}}_{\mathrm{w}} = \{u_0 \in X : u_0 = u(0)$ *for some* $u \in \bar{\mathcal{E}}((-\infty,\infty))\}$.
3. $\mathcal{A}_{\mathrm{w}}$ *is the maximal invariant and maximal quasi-invariant set w.r.t.* $\bar{\mathcal{E}}$.
4. *(Weak uniform tracking property) For any $\epsilon > 0$, there exists t_0, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0,\infty))$ satisfies $\mathrm{d}_{C([t^*,\infty);X_{\mathrm{w}})}(u,v) < \epsilon$, for some complete trajectory $v \in \bar{\mathcal{E}}((-\infty,\infty))$.*

Theorem 3.6. *Let $\mathcal{E}$ be an asymptotically compact evolutionary system. Then*

1. *The strong global attractor $\mathcal{A}_{\mathrm{s}}$ exists, it is strongly compact, and $\mathcal{A}_{\mathrm{s}} = \mathcal{A}_{\mathrm{w}}$.*

Furthermore, assume that $\mathcal{E}$ satisfies $\overline{\mathrm{A1}}$. Let $\bar{\mathcal{E}}$ be the closure of $\mathcal{E}$. Then

2. *(Strong uniform tracking property) For any $\epsilon > 0$ and $T > 0$, there exists t_0, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0,\infty))$ satisfies $\mathrm{d}_{\mathrm{s}}(u(t),v(t)) < \epsilon$, $\forall t \in [t^*, t^*+T]$, for some complete trajectory $v \in \bar{\mathcal{E}}((-\infty,\infty))$.*

Theorem 3.7. *Let $\mathcal{E}$ be an evolutionary system satisfying A1, A2, and A3, and assume that its closure $\bar{\mathcal{E}}$ satisfies $\bar{\mathcal{E}}((-\infty,\infty)) \subset C((-\infty,\infty);X_{\mathrm{s}})$. Then $\mathcal{E}$ is asymptotically compact.*

3.1. **Uniform w.r.t. symbol space global attractors.** In [CV94, CV02], Chepyzhov and Vishik studied the structure of the uniform (w.r.t. the initial time) global attractor of a process via that of the uniform (w.r.t. the symbol space) attractor of a family of processes with the symbol space being the strong closure of the translation family of the original symbol in an appropriate functional space. For further results in the case where the symbol space is a weak closure of the translation family of the original symbol we refer to [LWZ05, Lu06, Lu07]. In some cases (see e.g. open problems in [Lu07, CL09]) it is not clear how to choose a symbol space to obtain the structure of the uniform (w.r.t. the initial time) global attractor. Even though we solved this problem in Section 3 using a different approach, the uniform (w.r.t. the symbol space) attractor remains of mathematical interest. In this subsection we study this object and its relation to the uniform (w.r.t. the initial time) global attractor using our framework of evolutionary systems.

Definition 3.8. *Let $\mathcal{E}$ be an evolutionary system. If a map $\mathcal{E}^1$ that associates to each $I \in \mathcal{T}$ a subset $\mathcal{E}^1(I) \subset \mathcal{E}(I)$ is also an evolutionary system, we will call it an evolutionary subsystem of $\mathcal{E}$, and denote by $\mathcal{E}^1 \subset \mathcal{E}$.*

Let $\mathcal{E}_{\bar{\Sigma}}$ be an evolutionary system, and let $\Sigma \subset \bar{\Sigma}$ be such that $T(h)\Sigma = \Sigma$ for all $h \geq 0$. Then it is easy to check that $\mathcal{E}_{\Sigma}$ is also an evolutionary system. Hence, it is an evolutionary subsystem of $\mathcal{E}_{\bar{\Sigma}}$. For example, in Section 2.2, the evolutionary system defined by a process $\{U_{\sigma_0}(t,\tau)\}$ is an evolutionary subsystem of the evolutionary system defined by the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \bar{\Sigma}$, where $\bar{\Sigma}$ is the closure of the translation family Σ of the symbol σ_0 in some appropriate topological space.

Definition 3.9. *An evolutionary system $\mathcal{E}_\Sigma$ is a system with uniqueness if for every $u_0 \in X$ and $\sigma \in \Sigma$, there is a unique trajectory $u \in \mathcal{E}_\sigma([0,\infty))$ such that $u(0) = u_0$.*

Examples of evolutionary systems with uniqueness include the evolutionary systems defined previously by a process and a family of processes.

Theorem 3.10. *Let $\mathcal{E}_\Sigma$ be an evolutionary system with uniqueness and with symbol space Σ satisfying A1. Let $\bar{\Sigma}$ be the closure of Σ in some topology space $\Im$ and $\mathcal{E}_{\bar{\Sigma}} \supset \mathcal{E}_\Sigma$ be an evolutionary system with uniqueness satisfying $\bar{\text{A}}$1, and such that $u_n \in \mathcal{E}_{\sigma_n}([0,\infty))$, $u_n \to u$ in $C([0,\infty);X_{\mathrm{w}})$ and $\sigma_n \to \sigma$ in $\Im$ imply $u \in \mathcal{E}_\sigma([0,\infty))$. Then, their weak uniform global attractors $\mathcal{A}_{\mathrm{w}}^{\Sigma}$ and $\mathcal{A}_{\mathrm{w}}^{\bar{\Sigma}}$ are identical.*

Proof. Obviously, $\mathcal{A}_{\mathrm{w}}^{\Sigma}$ and $\mathcal{A}_{\mathrm{w}}^{\bar{\Sigma}}$ exist and $\mathcal{A}_{\mathrm{w}}^{\Sigma} \subset \mathcal{A}_{\mathrm{w}}^{\bar{\Sigma}}$. If there exists $x_0 \in \mathcal{A}_{\mathrm{w}}^{\bar{\Sigma}} \setminus \mathcal{A}_{\mathrm{w}}^{\Sigma}$, then there exist two disjoin balls $B_{\mathrm{w}}(\mathcal{A}_{\mathrm{w}}^{\Sigma}, \epsilon)$ and $B_{\mathrm{w}}(x_0, \epsilon)$. Since

$$\{u(t) | t \in \mathbb{R}, u \in \mathcal{E}((-\infty,\infty))\} \subset \mathcal{A}_{\mathrm{w}}^{\Sigma},$$

we can take, by Theorem 2.7, a complete trajectory $v(t) \in \mathcal{E}_\sigma((-\infty,\infty))$ with $\sigma \in \bar{\Sigma} \setminus \Sigma$ such that $v(0) = x_0$. The set $\{v(t)|t \in \mathbb{R}\}$ is d_{w}-attracted by $\mathcal{A}_{\mathrm{w}}^{\Sigma}$. Hence, there is some t_0 such that,

$$R_\Sigma(t_0)\{v(t)|t \in \mathbb{R}\} \subset B_{\mathrm{w}}(\mathcal{A}_{\mathrm{w}}^{\Sigma}, \epsilon).$$

Note that $\bar{\Sigma}$ is the closure of Σ in $\Im$. Take a sequence $\sigma_n \in \Sigma$ such that $\sigma_n \to \sigma$ in $\Im$. Consider a sequence of trajectories $u_n(t) \in \mathcal{E}_{\sigma_n}([0,\infty))$ satisfying $u_n(0) = v(-t_0)$ and

$$u_n(t_0) \in B_{\mathrm{w}}(\mathcal{A}_{\mathrm{w}}^{\Sigma}, \epsilon). \tag{17}$$

Thanks to $\bar{\text{A}}$1, $\{u_n(t)\}$ converges, passing to a subsequence and dropping a subindex, in $C([0,\infty);X_{\mathrm{w}})$, whose limit $u(t) \in \mathcal{E}_\sigma([0,\infty))$ due to $\sigma_n \to \sigma$ in $\Im$. By the fact that $u(0) = v(-t_0)$ and the uniqueness of the evolutionary system $\mathcal{E}_{\bar{\Sigma}}$, $u(t) = v(t - t_0)$, $t \geq 0$. However, (17) indicates that

$$x_0 = v(0) = u(t_0) \in \overline{B_{\mathrm{w}}(\mathcal{A}_{\mathrm{w}}^{\Sigma}, \epsilon)}^{\mathrm{w}}.$$

This is a contradiction. Hence, $\mathcal{A}_{\mathrm{w}}^{\Sigma} = \mathcal{A}_{\mathrm{w}}^{\bar{\Sigma}}$. □

Therefore, together with Theorems 3.5 and 2.7, Theorem 3.10 implies the follow:

Theorem 3.11. *Under the conditions of Theorem 3.10, let $\bar{\mathcal{E}}_\Sigma$ be the closure of the evolutionary system $\mathcal{E}_\Sigma$. Then the three weak uniform global attractors $\mathcal{A}_{\mathrm{w}}^{\Sigma}$, $\bar{\mathcal{A}}_{\mathrm{w}}^{\Sigma}$ and*

$\mathcal{A}_{\mathrm{w}}^{\bar{\Sigma}}$ *of the evolutionary systems* $\mathcal{E}_\Sigma$, $\bar{\mathcal{E}}_\Sigma$ *and* $\mathcal{E}_{\bar{\Sigma}}$, *respectively, are identical, and the following invariance property holds*

$$\begin{aligned}\mathcal{A}_{\mathrm{w}}^{\Sigma} &= \bar{\mathcal{A}}_{\mathrm{w}}^{\Sigma} = \mathcal{A}_{\mathrm{w}}^{\bar{\Sigma}} \\ &= \{u_0 : \ u_0 = u(0) \textit{ for some } u \in \bar{\mathcal{E}}((-\infty, \infty))\} \\ &= \{u_0 : \ u_0 = u(0) \textit{ for some } u \in \mathcal{E}_\Sigma((-\infty, \infty))\}.\end{aligned} \tag{18}$$

Moreover, the weak uniform tracking property holds.

Now, Theorem 3.3 ensures the strong compactness.

Theorem 3.12. *Under the conditions of Theorem 3.11, assume that* $\mathcal{E}_\Sigma$ *is asymptotically compact. Then the weak uniform global attractors in Theorem 3.11 are strongly compact strong uniform global attractors. Moreover, the strong uniform tracking property holds.*

In applications the auxiliary evolutionary system $\mathcal{E}_{\bar{\Sigma}}$ is usually asymptotically compact. For instance, in [Lu06], in the case of the 2D Navier-Stokes equations with non-slip boundary condition, $\bar{\Sigma}$ is taken as the closure of the translation family of a normal external force (see Section 5) in $L^{2,\mathrm{w}}_{\mathrm{loc}}(\mathbb{R}; V')$. Here, V' is the dual of the space of divergence-free vector fields with square-integrable derivatives and vanishing on the boundary, and $L^{2,\mathrm{w}}_{\mathrm{loc}}(\mathbb{R}; V')$ is the space $L^{2}_{\mathrm{loc}}(\mathbb{R}; V')$ endowed with local weak convergence topology. Then Theorem 3.12 applied to this system gives the strong uniform tracking property.

Similarly, together with Theorem 3.6 and 2.8, Theorem 3.10 implies the follows:

Theorem 3.13. *Under the conditions of Theorem 3.11, assume that* $\mathcal{E}_\Sigma$ *satisfies A2, A3 and the complete trajectories in (18) is strongly continuous. Then the weak uniform global attractor* $\mathcal{A}_{\mathrm{w}}^{\Sigma}$ *for* $\mathcal{E}_\Sigma$ *is strongly compact strong uniform global attractor. Moreover, the strong uniform tracking property holds.*

4. Trajectory attractor

A trajectory attractor for the 3D NSE was introduced in [Se96] and further studied in [CV97, CV02, SY02] by considering a family of auxiliary nonautonomous systems including the original system. In this section, with the results in preceding sections in hand, we will naturally construct a trajectory attractor for the original system under consideration, rather than for a family of systems. More precisely, we construct a trajectory attractor for the evolutionary system $\mathcal{E}$ satisfying A1 utilizing the trajectory attractor for its closure $\bar{\mathcal{E}}$, which is defined in [C09].

Let $\mathcal{F}^+ := C([0,\infty); X_{\mathrm{w}})$ and denote

$$\mathcal{K}^+ := \mathcal{E}([0\,\infty)) \subset \mathcal{F}^+.$$

Define the translation operators $T(s)$, $s \geq 0$,

$$(T(s)u)(t) := u(t+s)|_{[0,\infty)}, \quad u \in \mathcal{F}^+.$$

Due to the property 3 of the evolutionary system (see Definition 2.1 and 2.10), we have that,

$$T(s)\mathcal{K}^+ \subset \mathcal{K}^+, \quad \forall\, s \geq 0.$$

Note that $\mathcal{K}^+$ may not be closed, but is precompact in $\mathcal{F}^+$ due to A1. For a set $P \subset \mathcal{F}^+$ and $r > 0$ denote

$$B(P, r) := \{u \in \mathcal{F}^+ : \mathrm{d}_{C([0,\infty));X_\mathrm{w})}(u, P) < r\}.$$

A set $P \subset \mathcal{F}^+$ uniformly attracts a set $Q \subset \mathcal{K}^+$ if for any $\epsilon > 0$ there exists t_0, such that

$$T(t)Q \subset B(P, \epsilon), \quad \forall\, t \geq t_0.$$

Definition 4.1. *A set $P \subset \mathcal{F}^+$ is a trajectory attracting set for an evolutionary system $\mathcal{E}$ if it uniformly attracts $\mathcal{K}^+$.*

Definition 4.2. *A set $\mathfrak{A} \subset \mathcal{F}^+$ is a trajectory attractor for an evolutionary system $\mathcal{E}$ if $\mathfrak{A}$ is a minimal compact trajectory attracting set, and $T(t)\mathfrak{A} = \mathfrak{A}$ for all $t \geq 0$.*

It is easy to see that if a trajectory attractor exists, it is unique. Let $\bar{\mathcal{E}}$ be the closure of the evolutionary system $\mathcal{E}$ and let $\bar{\mathcal{K}} := \bar{\mathcal{E}}((-\infty, \infty))$ which is called the kernel of $\bar{\mathcal{E}}$. Let also

$$\Pi_+\bar{\mathcal{K}} := \{u(\cdot)|_{[0,\infty)} : u \in \bar{\mathcal{K}}\}.$$

Theorem 4.3. *Let $\mathcal{E}$ be an evolutionary system satisfying A1. Then the trajectory attractor exists and*

$$\mathfrak{A} = \Pi_+\bar{\mathcal{K}},$$

where $\bar{\mathcal{K}}$ is the kernel of the closure $\bar{\mathcal{E}}$ of the evolutionary system $\mathcal{E}$. Furthermore,

$$\mathcal{A}_\mathrm{w} = \mathfrak{A}(t) := \{u(t) : u \in \mathfrak{A}\}, \quad \forall\, t \geq 0.$$

Proof. Notice that Theorem 7.4 in [C09] states that the conclusions are valid for an evolutionary system satisfying A1.

Obviously, $\bar{\mathcal{E}}$ satisfies A1. Hence the trajectory attractor $\Pi_+\bar{\mathcal{K}}$ for $\bar{\mathcal{E}}$ uniformly attracts $\bar{\mathcal{K}}^+$. Now we verify that $\Pi_+\bar{\mathcal{K}}$ is a minimal trajectory attracting set for $\mathcal{E}$. Assume that there exists a compact trajectory attracting set P strictly included in $\Pi_+\bar{\mathcal{K}}$. Then there exist $\epsilon > 0$ and

$$u \in \Pi_+\bar{\mathcal{K}} \setminus B(P, 2\epsilon).$$

Let $v \in \bar{\mathcal{E}}((-\infty, \infty))$ be such that $v|_{[0,\infty)} = u$. Let also $v_n(\cdot) = v(\cdot - n)|_{[0,\infty)}$. Note that $v_n \in \bar{\mathcal{E}}([0, \infty))$ and

$$T(n)v_n = u \notin B(P, 2\epsilon), \quad \forall\, n.$$

Now take $u_n \in \mathcal{E}([0, \infty))$ such that

$$\mathrm{d}_{C([0,\infty));X_\mathrm{w})}(u_n, v_n) < \epsilon/2^n, \quad \forall\, n.$$

By the definition of the metric $\mathrm{d}_{C([0,\infty));X_{\mathrm{w}})}$, we have

$$\mathrm{d}_{C([0,\infty));X_{\mathrm{w}})}(T(n)u_n, T(n)v_n) \leq 2^n \mathrm{d}_{C([0,\infty));X_{\mathrm{w}})}(u_n, v_n), \quad \forall\, n.$$

Hence,

$$\mathrm{d}_{C([0,\infty));X_{\mathrm{w}})}(T(n)u_n, u) < \epsilon, \quad \forall\, n,$$

which implies that

$$T(n)u_n \notin B(P,\epsilon), \quad \forall\, n.$$

Therefore, P is not a trajectory attracting set for $\mathcal{E}$, which is a contradiction. □

Furthermore, the asymptotical compactness of $\mathcal{E}$ implies a uniform strong convergence of solutions toward the trajectory attractor.

Theorem 4.4. *Let $\mathcal{E}$ be an asymptotically comapct evolutionary system satisfying A1. Then the trajectory attractor $\mathfrak{A}$ uniformly attracts $\mathcal{K}^+$ in $L^\infty_{\mathrm{loc}}((0,\infty))$.*

Proof. This is just a consequence of Theorem 3.3. □

Finally, by the strong continuity of the complete trajectories, we have the following.

Theorem 4.5. *Let $\mathcal{E}$ be an evolutionary system satisfying A1, A2 and A3. If $\mathfrak{A} \subset C([0,\infty); X_{\mathrm{s}})$, then the trajectory attractor $\mathfrak{A}$ uniformly attracts $\mathcal{K}^+$ in $L^\infty_{\mathrm{loc}}((0,\infty))$.*

Proof. Since $\mathfrak{A} \subset C([0,\infty); X_{\mathrm{s}})$, Theorem 3.7 implies that the evolutionary system $\mathcal{E}$ is asymptotically compact. Therefore, Theorem 4.4 yields that $\mathfrak{A}$ uniformly attracts $\mathcal{K}^+$ in $L^\infty_{\mathrm{loc}}((0,\infty))$. □

5. 3D NAVIER-STOKES EQUATION

Consider the space periodic 3D incompressible Navier-Stokes equations (NSE)

$$\begin{cases} \dfrac{d}{dt}u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(t), \\ \nabla\cdot u = 0, \end{cases} \tag{19}$$

where u, the velocity, and p, the pressure, are unknowns; $f(t)$ is a given driving force, and $\nu > 0$ is the kinematic viscosity coefficient of the fluid. By a Galilean change of variables, we can assume that the space average of u is zero, i.e.,

$$\int_\Omega u(x,t)\,dx = 0, \qquad \forall t,$$

where $\Omega = [0,L]^3$ is a periodic box.[1]

First, let us introduce some notations and functional setting. Denote by $(\cdot,\cdot)$ and $|\cdot|$ the $L^2(\Omega)^3$-inner product and the corresponding $L^2(\Omega)^3$-norm. Let $\mathcal{V}$ be the

[1]The no-slip case can be considered in a similar way, only with some adaption on the functional setting.

space of all $\mathbb{R}^3$ trigonometric polynomials of period L in each variable satisfying $\nabla \cdot u = 0$ and $\int_\Omega u(x)\,dx = 0$. Let H and V to be the closures of $\mathcal{V}$ in $L^2(\Omega)^3$ and $H^1(\Omega)^3$, respectively. Define the strong and weak distances by

$$\mathrm{d_s}(u,v) := |u-v|, \qquad \mathrm{d_w}(u,v) = \sum_{\kappa\in\mathbb{Z}^3} \frac{1}{2^{|\kappa|}} \frac{|u_\kappa - v_\kappa|}{1+|u_\kappa - v_\kappa|}, \qquad u,v \in H,$$

where u_κ and v_κ are Fourier coefficients of u and v respectively. Note that the weak metric $\mathrm{d_w}$ induces the weak topology in any ball in $L^2(\Omega)^3$.

Let also $P_\sigma : L^2(\Omega)^3 \to H$ be the L^2-orthogonal projection, referred to as the Leray projector. Denote by $A = -P_\sigma\Delta = -\Delta$ the Stokes operator with the domain $D(A) = (H^2(\Omega))^3 \cap V$. The Stokes operator is a self-adjoint positive operator with a compact inverse. Let

$$\|u\| := |A^{1/2}u|,$$

which is called the enstrophy norm. Note that $\|u\|$ is equivalent to the H^1-norm of u for $u \in D(A^{1/2})$.

Now denote $B(u,v) := P_\sigma(u\cdot\nabla v) \in V'$ for all $u, v \in V$. This bilinear form has the following property:

$$\langle B(u,v), w\rangle = -\langle B(u,w), v\rangle, \qquad u,v,w \in V,$$

in particular, $\langle B(u,v), v\rangle = 0$ for all $u, v \in V$.

Now we can rewrite (19) as the following differential equation in V':

$$\frac{d}{dt}u + \nu Au + B(u,u) = g, \tag{20}$$

where u is a V-valued function of time and $g = P_\sigma f$.

Definition 5.1. *A weak solution of* (19) *on* $[T,\infty)$ *(or* $(-\infty,\infty)$*, if* $T=-\infty$*) is an* H*-valued function* $u(t)$ *defined for* $t \in [T,\infty)$*, such that*

$$\frac{d}{dt}u \in L^1_{\mathrm{loc}}([T,\infty);V'), \qquad u(t) \in C([T,\infty);H_{\mathrm{w}}) \cap L^2_{\mathrm{loc}}([T,\infty);V),$$

and

$$(u(t)-u(t_0), v) = \int_{t_0}^t \left(-\nu((u,v)) - \langle B(u,u),v\rangle + \langle g,v\rangle\right) ds, \tag{21}$$

for all $v \in V$ *and* $T \le t_0 \le t$.

Theorem 5.2 (Leray, Hopf)**.** *For every* $u_0 \in H$ *and* $g \in L^2_{\mathrm{loc}}(\mathbb{R};V')$*, there exists a weak solution of (19) on* $[T,\infty)$ *with* $u(T) = u_0$ *satisfying the following energy inequality*

$$|u(t)|^2 + 2\nu\int_{t_0}^t \|u(s)\|^2\,ds \le |u(t_0)|^2 + 2\int_{t_0}^t \langle g(s), u(s)\rangle\,ds \tag{22}$$

for all $t \geq t_0$, t_0 a.e. in $[T, \infty)$.

Definition 5.3. *A Leray-Hopf solution of* (19) *on the interval $[T, \infty)$ is a weak solution on $[T, \infty)$ satisfying the energy inequality (22) for all $T \leq t_0 \leq t$, t_0 a.e. in $[T, \infty)$. The set Ex of measure 0 on which the energy inequality does not hold will be called the exceptional set.*

Now fixed an external force g_0 that is translational bounded in $L^2_{\mathrm{loc}}(\mathbb{R}; V')$, i.e.,

$$\|g_0\|^2_{L^2_{\mathrm{b}}} := \sup_{t \in \mathbb{R}} \int_t^{t+1} \|g_0(s)\|^2_{V'} \, ds < \infty.$$

Then g_0 is translation compact in $L^{2,\mathrm{w}}_{\mathrm{loc}}(\mathbb{R}; V')$, i.e., the translation family $\Sigma := \{g_0(\cdot + h) | h \in \mathbb{R}\}$ of g_0 is precompact in $L^{2,\mathrm{w}}_{\mathrm{loc}}(\mathbb{R}; V')$. Note that,

$$\|g\|^2_{L^2_{\mathrm{b}}} \leq \|g_0\|^2_{L^2_{\mathrm{b}}}, \quad \forall\, g \in \Sigma. \tag{23}$$

Due to the energy inequality (22) we have

$$|u(t)|^2 + \nu \int_{t_0}^t \|u(s)\|^2 \, ds \leq |u(t_0)|^2 + \frac{1}{\nu} \int_{t_0}^t \|g(s)\|^2_{V'} \, ds, \qquad \forall g \in \Sigma,$$

for all $t \geq t_0$, t_0 a.e. in $[T, \infty)$. Here $u(t)$ is a Leray-Hopf solutions of (19) with force g on $[T, \infty)$. By Gronwall's inequality there exists an absorbing ball $B_{\mathrm{s}}(0, R)$, where the radius R depends on L, ν, and $\|g_0\|^2_{L^2_{\mathrm{b}}}$. Let X be a closed absorbing ball

$$X = \{u \in H : |u| \leq R\},$$

which is also weakly compact. Then for any bounded set $A \subset H$, there exists a time $t_1 \geq T$, such that

$$u(t) \in X, \qquad \forall t \geq t_1,$$

for every Leray-Hopf solution $u(t)$ with the force $g \in \Sigma$ and the initial data $u(T) \in A$. For any sequence of Leray–Hopf solutions u_n the following result holds.

Lemma 5.4. *Let $u_n(t)$ be a sequence of Leray-Hopf solutions of* (19) *with forces $g_n \in \Sigma$, such that $u_n(t) \in X$ for all $t \geq t_1$. Then*

$$\begin{aligned} &u_n \ \textit{ is bounded in } \ L^2(t_1, t_2; V) \ \textit{ and } \ L^\infty(t_1, t_2; H), \\ &\frac{d}{dt} u_n \ \textit{ is bounded in } \ L^{4/3}(t_1, t_2; V'), \end{aligned} \tag{24}$$

for all $t_2 > t_1$. Moreover, there exists a subsequence n_j converges to some $u(t)$ in $C([t_1, t_2]; H_{\mathrm{w}})$, i.e.,

$$(u_{n_j}, v) \to (u, v) \qquad \textit{uniformly on} \qquad [t_1, t_2],$$

as $n_j \to \infty$, for all $v \in H$.

Proof. The proof is standard (see eg. [CF89, Ro01]). Here we just sketch some steps. Take a sequence u_n satisfying (19) with forces g_n. By (20), we have

$$\frac{d}{dt}u_n + \nu A u_n + B(u_n, u_n) = g_n, \tag{25}$$

Classical estimates imply the boundedness in (24). Then, passing to a subsequence and dropping a subindex, we can obtain that

$$\begin{aligned} u_n \to u \quad & \text{weak-star in } L^\infty(t_1, t_2; H), \\ & \text{weakly in } L^2(t_1, t_2; V), \\ & \text{strongly in } L^2(t_1, t_2; H), \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}u_n \to \frac{d}{dt}u \quad & \text{weakly in } L^{4/3}(t_1, t_2; V'), \\ Au_n \to Au \quad & \text{weakly in } L^2(t_1, t_2; V'), \\ B(u_n, u_n) \to B(u, u) \quad & \text{weakly in } L^{4/3}\left(t_1, t_2; V'\right), \end{aligned}$$

for some

$$u \in L^\infty(t_1, t_2; H) \cap L^2(t_1, t_2; V).$$

Again, passing to a subsequence and dropping a subindex, we also have,

$$g_n \to g \qquad \text{weakly in } L^2(t_1, t_2; V'), \tag{26}$$

with $g \in L^2(t_1, t_2; V')$. Passing to the limit yields

$$\frac{d}{dt}u + \nu A u + B(u, u) = g.$$

It follows from (21) that $u_n \to u$ in $C([t_1, t_2]; H_{\mathrm{w}})$. □

Remark 5.5. *In the autonomous case, i.e., $f(t)$ is independent of t, the limit u is a Leray-Holf solution. However, we don't know here whether it is a Leray-Holf solution yet.*

Consider an evolutionary system for which a family of trajectories consists of all Leray-Hopf solutions of the 3D Navier-Stokes equations with a fixed force g_0 in X. More precisely, define

$$\begin{aligned} \mathcal{E}([T,\infty)) := \{u(\cdot) : u(\cdot) & \text{ is a Leray-Hopf solution on } [T,\infty) \\ & \text{with the force } g \in \Sigma \text{ and } u(t) \in X,\ \forall t \in [T,\infty)\}, \qquad T \in \mathbb{R}, \\ \mathcal{E}((-\infty,\infty)) := \{u(\cdot) : u(\cdot) & \text{ is a Leray-Hopf solution on } (-\infty,\infty) \\ & \text{with the force } g \in \Sigma \text{ and } u(t) \in X,\ \forall t \in (-\infty,\infty)\}. \end{aligned}$$

Clearly, the properties 1–4 of $\mathcal{E}$ hold, if we utilize the translation semigroup $\{T(s)\}_{s\geq 0}$. Therefore, thanks to Theorem 3.5, the uniform weak global attractor $\mathcal{A}_{\mathrm{w}}$ for this evolutionary system exists.

Now we give the definition of normal function which was first put forward in [LWZ05].

Definition 5.6. *Let $\mathcal{B}$ be a Banach space. A function $\varphi(s) \in L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{B})$ is said to be normal in $L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{B})$ if for any $\epsilon > 0$, there exists $\delta > 0$, such that*

$$\sup_{t\in\mathbb{R}} \int_t^{t+\delta} \|\varphi(s)\|^2_{\mathcal{B}}\, ds \leq \epsilon.$$

Note that the class of normal functions is a proper closed subspace of the class of translation bounded functions (see [LWZ05] for more details). Then, we have the following.

Lemma 5.7. *The evolutionary system $\mathcal{E}$ of the 3D NSE with the force g_0 satisfies A1 and A3. Moreover, if g_0 is normal in $L^2_{\mathrm{loc}}(\mathbb{R};V')$ then A2 holds.*

Proof. First note that $\mathcal{E}([0,\infty)) \subset C([0,\infty);H_{\mathrm{w}})$ by the definition of a Leray-Hopf solution. Now take any sequence $u_n \in \mathcal{E}([0,\infty))$, $n = 1,2,\ldots$. Thanks to Lemma 5.4, there exists a subsequence, still denoted by u_n, that converges to some $u^1 \in C([0,1];H_{\mathrm{w}})$ in $C([0,1];H_{\mathrm{w}})$ as $n\to\infty$. Passing to a subsequence and dropping a subindex once more, we obtain that $u_n \to u^2$ in $C([0,2];H_{\mathrm{w}})$ as $n\to\infty$ for some $u^2 \in C([0,2];H_{\mathrm{w}})$. Note that $u^1(t) = u^2(t)$ on $[0,1]$. Continuing this diagonalization process, we obtain a subsequence u_{n_j} of u_n that converges to some $u \in C([0,\infty);H_{\mathrm{w}})$ in $C([0,\infty);H_{\mathrm{w}})$ as $n_j\to\infty$. Therefore, A1 holds.

Let now $u_n \in \mathcal{E}([0,\infty))$ be such that $u_n \to u \in C([0,T];H_{\mathrm{w}})$ in $C([0,T];H_{\mathrm{w}})$ as $n\to\infty$ for some $T>0$. Thanks to Lemma 5.4 again, the sequence $\{u_n\}$ is bounded in $L^2([0,T];V)$. Hence,

$$\int_0^T |u_n(s)-u(s)|^2\, ds \to 0, \qquad \text{as} \qquad n\to\infty.$$

In particular, $|u_n(t)| \to |u(t)|$ as $n\to\infty$ a.e. on $[0,T]$, i.e., A3 holds.

Now assume that g_0 is normal in $L^2_{\mathrm{loc}}(\mathbb{R};V')$. Then given $\epsilon>0$, there exists $\delta>0$, such that

$$\sup_{t\in\mathbb{R}} \int_{t-\delta}^{t} \|g_0(s)\|^2_{V'}\, ds \leq \nu\epsilon.$$

Take any $u \in \mathcal{E}([0,\infty))$ and $t>0$. Since $u(t)$ is a Leray-Hopf solution, it satisfies the energy inequality (22)

$$|u(t)|^2 + 2\nu\int_{t_0}^t \|u(s)\|^2\, ds \leq |u(t_0)|^2 + 2\int_{t_0}^t \langle g(s),u(s)\rangle\, ds,$$

for all $0 \le t_0 \le t$, $t_0 \in [0,\infty) \setminus Ex$, where Ex is a set of zero measure. Hence, together with (23),

$$\begin{aligned}|u(t)|^2 &\le |u(t_0)|^2 + \frac{1}{\nu}\int_{t_0}^{t} \|g_0\|_{V'}^2 \, ds \\ &\le |u(t_0)|^2 + \epsilon,\end{aligned}$$

for all $t_0 \ge 0$, such that $t_0 \in (t-\delta, t) \setminus Ex$. Therefore, A2 holds. □

Now Lemma 5.7, Theorem 3.5 and 3.6 yield the following.

Theorem 5.8. *The uniform weak global attractor $\mathcal{A}_{\mathrm{w}}$ for the 3D NSE with force g_0 exists, $\mathcal{A}_{\mathrm{w}}$ is the maximal invariant and maximal quasi-invariant set w.r.t. the closure $\bar{\mathcal{E}}$ of the corresponding evolutionary system $\mathcal{E}$, and*

$$\mathcal{A}_{\mathrm{w}} = \omega_{\mathrm{w}}(X) = \omega_{\mathrm{s}}(X) = \{u(0) : u \in \bar{\mathcal{E}}((-\infty,\infty))\}.$$

Moreover, the weak uniform tracking property holds

Theorem 5.9. *If g_0 is normal in $L^2_{\mathrm{loc}}(\mathbb{R}; V')$ and every complete trajectory of $\bar{\mathcal{E}}$ is strongly continuous, then the weak global attractor $\mathcal{A}_{\mathrm{w}}$ is a strongly compact strong global attractor $\mathcal{A}_{\mathrm{s}}$. Moreover, the strong uniform tracking property holds.*

Finally, we obtain the trajectory attractor for 3D NSE with a fixed time-dependent force g_0 due to Theorem 4.3 and 4.5.

Theorem 5.10. *The trajectory attractor for 3D NSE with force g_0 exists and*

$$\mathfrak{A} = \Pi_+\bar{\mathcal{E}}((-\infty,\infty)) = \{u(\cdot)|_{[0,\infty)} : u \in \bar{\mathcal{E}}((-\infty,\infty))\},$$

satisfying

$$\mathcal{A}_{\mathrm{w}} = \mathfrak{A}(t) = \{u(t) : u \in \mathfrak{A}\}, \quad \forall\, t \ge 0.$$

Furthermore, if g_0 is normal in $L^2_{\mathrm{loc}}(\mathbb{R}, V')$ and every complete trajectory of $\bar{\mathcal{E}}$ is strongly continuous then the trajectory attractor $\mathfrak{A}$ uniformly attracts $\mathcal{E}([0,\infty))$ in $L^\infty_{\mathrm{loc}}((0,\infty))$.

6. OPEN PROBLEMS

In this section we assume that g_0 is translation compact in $L^2_{\mathrm{loc}}(\mathbb{R}; V')$ and denote by $\bar{\Sigma}$ the closure of Σ in $L^2_{\mathrm{loc}}(\mathbb{R}; V')$. Note that the class of translation compact functions is also a closed subspace of the class of translation bounded functions, but it is a proper subset of the class of normal functions (for more details, see [LWZ05]). Note that the argument in Section 5 before Lemma 5.4 is valid for Σ replaced by $\bar{\Sigma}$ and Lemma 5.4 can be improved as follows.

Lemma 6.1. *Let $u_n(t)$ be a sequence of Leray-Hopf solutions of* (19) *with forces $g_n \in \bar{\Sigma}$, such that $u_n(t) \in X$ for all $t \geq t_1$. Then*

$$\begin{aligned} u_n \ \text{ is bounded in } \ & L^2(t_1,t_2;V) \ \text{ and } \ L^\infty(t_1,t_2;H), \\ \frac{d}{dt}u_n \ \text{ is bounded in } \ & L^{4/3}(t_1,t_2;V'), \end{aligned}$$

for all $t_2 > t_1$. Moreover, there exists a subsequence n_j, such that g_{n_j} converges in $L^{2,\mathrm{w}}_{\mathrm{loc}}(\mathbb{R};V')$ to some $g \in \bar{\Sigma}$ and u_{n_j} converges in $C([t_1,t_2];H_{\mathrm{w}})$ to some Leray-Hopf solution $u(t)$ of (19) *with the force g, i.e.,*

$$(u_{n_j}, v) \to (u, v) \qquad \text{uniformly on} \qquad [t_1, t_2],$$

as $n_j \to \infty$, for all $v \in H$.

Proof. See [CF89, CV02]. Here we give a brief sketch.

The proof of Lemma 5.4 is still valid if we substitute $\bar{\Sigma}$ for Σ. So, the remains is to verify (22) for the limit u. We have

$$|u_n(t)|^2 + 2\nu \int_{t_0}^t \|u_n(s)\|^2\, ds \leq |u_n(t_0)|^2 + 2\int_{t_0}^t \langle g_n(s), u_n(s)\rangle\, ds \tag{27}$$

for all $t \geq t_0$, t_0 a.e. in $[t_1,\infty)$. Note that

$$\begin{aligned} u_n(t) \to u(t) \quad & \text{weakly in } H, \quad \forall\, t \geq t_1, \\ & \text{strongly in } H, \quad t \text{ a.e. in } [t_1,\infty), \\ & \text{weakly in } L^2_{\mathrm{loc}}(t_1,\infty;V), \end{aligned}$$

and the convergence in (26) is strong for g_0 is translation compact in $L^2_{\mathrm{loc}}(\mathbb{R};V')$. Therefore, taking the limit of (27) we obtain the energy inequality

$$|u(t)|^2 + 2\nu \int_{t_0}^t \|u(s)\|^2\, ds \leq |u(t_0)|^2 + 2\int_{t_0}^t \langle g(s), u(s)\rangle\, ds$$

for all $t \geq t_0$, t_0 a.e. in $[t_1,\infty)$. □

Due to this lemma, now we can consider another evolutionary system with $\bar{\Sigma}$ as a symbol space. The family of trajectories of the evolutionary system consists of all Leray-Hopf solutions of the family of 3D Navier-Stokes equations with forces $g \in \bar{\Sigma}$ in X:

$$\begin{aligned} \mathcal{E}_{\bar{\Sigma}}([T,\infty)) := \{u(\cdot) : u(\cdot) & \text{ is a Leray-Hopf solution on } [T,\infty) \\ & \text{with the force } g \in \bar{\Sigma} \text{ and } u(t) \in X,\ \forall t \in [T,\infty)\}, \qquad T \in \mathbb{R}, \\ \mathcal{E}_{\bar{\Sigma}}((-\infty,\infty)) := \{u(\cdot) : u(\cdot) & \text{ is a Leray-Hopf solution on } (-\infty,\infty) \\ & \text{with the force } g \in \bar{\Sigma} \text{ and } u(t) \in X,\ \forall t \in (-\infty,\infty)\}. \end{aligned}$$

Obviously, $\mathcal{E} \subset \mathcal{E}_{\bar{\Sigma}}$.

We have the following lemma.

Lemma 6.2. *The evolutionary system $\mathcal{E}_{\bar{\Sigma}}$ of the family of 3D NSE with forces in $\bar{\Sigma}$ satisfies $\bar{A}1$, $\bar{A}2$ and $\bar{A}3$.*

Proof. The proof is similar to Lemma 5.7. The difference is that we have to use Lemma 6.1 instead of Lemma 5.4, and that $\{u^i\}$ and u would now be contained in $\mathcal{E}_{\bar{\Sigma}}([0,\infty))$. □

Similarly, Lemma 6.2 and Theorem 2.7 yield the following (cf. [CV02]).

Theorem 6.3. *The uniform weak global attractor $\mathcal{A}_{\mathrm{w}}^{\bar{\Sigma}}$ for the family of 3D NSE with forces $g \in \bar{\Sigma}$ exists, $\mathcal{A}_{\mathrm{w}}^{\bar{\Sigma}}$ is the maximal invariant and maximal quasi-invariant set w.r.t. the corresponding evolutionary system $\mathcal{E}_{\bar{\Sigma}}$, and*

$$\mathcal{A}_{\mathrm{w}}^{\bar{\Sigma}} = \{u(0) : u \in \mathcal{E}_{\bar{\Sigma}}((-\infty,\infty))\}.$$

Moreover, the weak uniform tracking property holds

Theorem 2.8 gives a criterion for strong compactness of the attractor.

Theorem 6.4. *If every complete trajectory of the family of 3D NSE with forces $g \in \bar{\Sigma}$ is strongly continuous, then the weak global attractor $\mathcal{A}_{\mathrm{w}}^{\bar{\Sigma}}$ is a strongly compact strong global attractor $\mathcal{A}_{\mathrm{s}}^{\bar{\Sigma}}$. Moreover, the strong uniform tracking property holds.*

Let $\bar{\mathcal{E}}$ be the closure of the evolutionary system $\mathcal{E}$. Obviously, $\mathcal{E} \subset \bar{\mathcal{E}} \subset \mathcal{E}_{\bar{\Sigma}}$. Then, an interesting problem arises:

Open Problem 6.5. *Are the uniform global attractors $\mathcal{A}_{\bullet}$ and $\mathcal{A}_{\bullet}^{\bar{\Sigma}}$ in Theorems 5.8 and 6.4 identical?*

If the solutions of 3D NSE are unique, then the answer is positive due to Theorem 3.11 and 3.12. However, the negative answer, i.e., $\mathcal{A}_{\bullet} \subsetneq \mathcal{A}_{\bullet}^{\bar{\Sigma}}$, would imply that the Leray-Hopf weak solutions are not unique and the uniform (w.r.t. symbol space) attractor doesn't satisfy the minimality property with respect to uniform (w.r.t. initial time) attracting for the original 3D NSE with fixed external force g_0.

We can also obtain a trajectory attractor of $\mathcal{E}_{\bar{\Sigma}}$ as in Section 5:

Theorem 6.6. *The trajectory attractor for the family of 3D NSE with forces $g \in \bar{\Sigma}$ exists and*

$$\mathfrak{A}^{\bar{\Sigma}} = \Pi_+ \mathcal{E}_{\bar{\Sigma}}((-\infty,\infty)) = \{u(\cdot)|_{[0,\infty)} : u \in \mathcal{E}_{\bar{\Sigma}}((-\infty,\infty))\},$$

satisfying

$$\mathcal{A}_{\mathrm{w}}^{\bar{\Sigma}} = \mathfrak{A}^{\bar{\Sigma}}(t) = \{u(t) : u \in \mathfrak{A}^{\bar{\Sigma}}\}, \quad \forall\, t \geq 0.$$

Furthermore, if every complete trajectory of $\mathcal{E}_{\bar{\Sigma}}$ is strongly continuous then $\mathfrak{A}^{\bar{\Sigma}}$ uniformly attracts $\mathcal{E}_{\bar{\Sigma}}([0,\infty))$ in $L^{\infty}_{\mathrm{loc}}((0,\infty))$.

A similar problem on the relationship of this trajectory attractor and that of $\mathcal{E}$ also arises:

Open Problem 6.7. *Are the trajectory attractors $\mathfrak{A}$ and $\mathfrak{A}^{\bar{\Sigma}}$ in Theorem 5.10 and 6.6 identical?*

This open problem hints that, in general, the trajectory attractors constructed in [CV02] for the systems without uniqueness might not satisfy the minimality property.

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