Turbulent boundary layer equations
Équations de la couche limite turbulente

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Abstract

We study a boundary layer problem for the Navier-Stokes-alpha model obtaining a generalization of the Prandtl equations which we conjecture to represent the averaged flow in a turbulent boundary layer. We study the equations for the semi-infinite plate, both theoretically and numerically. Solutions agree with some experimental data in a part of the turbulent boundary layer.

boundary layer / turbulence / α-model

Résumé

Nous considérons le problème de la couche limite pour l'α-modèle des équations de Navier-Stokes, obtenant d'abord une généralisation des équations de Prandtl. Notre hypothèse est que les solutions de ces équations représentent l'écoulement moyen dans une partie de la couche limite turbulente. Nous étudions, analytiquement et numériquement, ces solutions pour la plaque plane semi-infinie. Les solutions numériques donnent une très bonne approximation des certaines données expérimentales dans la couche limite turbulente.

couche limite / turbulence / α-modèle

1 Version française abrégée

La théorie de la couche limite, initiée par L. Prandtl en 1904, a maintenant des nombreuses applications, surtout dans l'aérodynamique. Toutefois, l'écoulement turbulent dans une couche limite n'est pas encore bien compris. Récemment un α-modèle des équations d'Euler a été introduit dans [1] comme une généralisation aux équations d'Euler de l'équation de Camassa-Holm en dimension 1 [2]. Ensuite l'α-modèle des équations de Navier-Stokes (voir (3)) a été utilisé dans l'étude des moyennes des écoulements turbulents. Précisément ce modèle a été proposé comme une approximation des équations de Reynolds et ses solutions ont été comparées avec les données expérimentales et numériques des écoulements turbulents dans une conduite [3]-[5].

Dans cette Note nous obtenons (par une voie similaire à celle de Prandtl) des équations pour les écoulements turbulents dans une couche limite, qui généralisent les équations de Prandtl et qui constituent essentiellement un α-modèle de ces derniers équations (voir (4)). Dans le cas d'une plaque plane semi-infinie, la longueur α dans notre modèle est proportionnelle avec l'épaisseur de la couche limite. En utilisant la représentation de Blasius [6] nous réduisons notre système d'équations aux dérivée partielles à l'équation différentielle (5), qui peut s'écrire aussi sous la forme

$$m'''(\xi) + \frac{1}{2}h(\xi)m''(\xi) = 0 \quad (0 \leq \xi < \infty), \tag{1}$$

où $m = h - \beta h''$ et β est une paramètre non dimensionnel. Les solutions aux limites sont $h(0) = h'(0) = 0$ et $h'(\xi) \to 1$ pour $\xi \to \infty$. L'écoulement moyen est donné par les formules de Blasius en fonction de h'. Notre résultat théorique principal est le suivant (voir Théorème 1): *Pour tout $a > 0$, b il existe $c = c(a, b)$ tel que $h'(\xi) \to \gamma \geq 0$ pour $\xi \to \infty$, où h est une solution de (1) satisfaisant les conditions aux limites $h(0) = 0$, $h'(0) = 0$, $h''(0) = a$, $h'''(0) = b$ et $h''''(0) = c$.* En exceptant le cas non-générique $\gamma = 0$, la condition $h'(\xi) \to 1$ pour $\xi \to \infty$ s'obtient de la solution donnée par le théorème ci-dessus par un changement d'échelle.

L'équation classique de Blasius est le cas particulier de (1) où $\beta = 0$. Dans ce cas, la solution classique donne une bonne approximation des données expérimentales dans la région laminaire de la couche limite. Les solutions de (1) pour $\beta > 0$ donnent une très bonne approximation des données expérimentales dans un intervalle plus large des nombres de Reynolds locaux. Cet intervalle contient aussi la région de transition et une partie de la région turbulente de la couche limite (vous Fig. 1). Pour des nombres de Reynolds plus grands cette approximation ne tient plus. La description mathématique de l'écoulement dans ce cas extrême est encore un problème ouvert.

2 Introduction

Boundary-layer theory, first introduced by L. Prandtl in 1904, is now fundamental to many applications in fluid mechanics, especially in aerodynamics. However, the turbulent boundary layer flow is not well understood yet.

Recently, an Euler-α model has been introduced in [2] as a generalization to n dimensions of the one-dimensional Camassa-Holm equation that describes shallow water waves [1]. Henceforth, the Navier-Stokes-alpha model of fluid turbulence (NS-α model) was used to study the averaged velocity field of a turbulent fluid. This model was proposed as a closure approximation for the Reynolds equation, and it's solutions were compared with empirical data for turbulent flows in channels and pipes [3]-[5].

We use NS-α model to study part of the turbulent boundary layer. Similarly to the Prandtl's derivation in the laminar case, we obtain turbulent boundary layer equations that generalize Prandtl equations. In the case of semi-infinite plate using Blasius's similarity variable and assuming that α in NS-α model is proportional to the thickness of boundary layer, we reduce those equations to the following ordinary differential equation:

$$m''' + \frac{1}{2}hm'' = 0, \tag{2}$$

where $m = h - \beta^2 h''$ for some parameter β. The boundary conditions are $h(0) = h'(0) = 0$, and $h'(\xi) \to 1$ as $\xi \to \infty$. Velocity components can be obtained by scaling of h'.

Observe that Blasius equation is a particular case of (2) when $\beta = 0$. The solution of that equation matches experimental data in the laminar region of the boundary layer. Solutions of (2) match experimental data for a larger interval of local Reynolds numbers, that includes the transition region and a part of the turbulent boundary layer region. For higher Reynolds numbers there is no good match with experimental data. The mathematical description of the flow in that extreme case remains an open problem.

3 Derivation

We study two-dimensional steady incompressible viscous flow near a flat surface. Let x be the coordinate along the surface, y be the coordinate normal to the surface. Let also (u, v) be the velocity of the flow.

Two-dimensional Navier-Stokes-α model is used to study a boundary layer flow.

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{v} + (\mathbf{u} \cdot \nabla)\mathbf{v} + v_j \nabla u_j = \nu \Delta \mathbf{v} - \nabla q \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \tag{3}$$

where $\mathbf{u} = (u, v)$, and $\mathbf{v} = (\gamma, \tau)$ is a momentum defined in the following way:

$$\mathbf{v} = \mathbf{u} - \frac{\partial}{\partial x_i}\left(\alpha^2 \delta_{ij} \frac{\partial}{\partial x_j} \mathbf{u}\right).$$

Non-slip boundary conditions are used: $\mathbf{u}|_{y=0} = 0$. The other boundary conditions are going to be determined later.

Fix l on the x-axis and define $\epsilon(l)$ to be $\epsilon := 1/\sqrt{R_l} = \sqrt{\nu/(u_e l)}$, where ν is viscosity and u_e is the horizontal velocity component of the external flow. We change variables $x_1 = \frac{1}{l}x$, $y_1 = \frac{1}{\epsilon l}y$, $u_1 = \frac{1}{u_e}u$, $v_1 = \frac{1}{\epsilon u_e}v$, $q_1 = \frac{1}{u_e^2}q$, $\alpha_1 = \frac{1}{\epsilon l}\alpha$. In addition, assume that α_1 doesn't depend on y variable. After neglecting terms with high powers of ϵ, dropping subscripts and denoting

$$w = \left(1 - \alpha^2 \frac{\partial^2}{\partial y^2}\right) u,$$

we derive the following turbulent boundary layer equations that generalize Prandtl equations:

$$\begin{cases} u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial x}u = \frac{\partial^2}{\partial y^2}w - \frac{\partial}{\partial x}q \\ w\frac{\partial}{\partial y}u = -\frac{\partial}{\partial y}q \\ \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0. \end{cases} \tag{4}$$

Note that derivative of q is not zero in y-direction. Therefore, we introduce $Q(x, y) := q + \frac{1}{2}u^2 - \frac{1}{2}\alpha^2\left(\frac{\partial}{\partial y}u\right)^2$, and obtain the following equations:

$$\begin{cases} u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + \alpha^2\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right) + \frac{1}{2}\frac{\partial}{\partial x}\alpha^2 \cdot \left(\frac{\partial}{\partial y}u\right)^2 = \frac{\partial^2}{\partial y^2}w - \frac{\partial}{\partial x}Q \\ \frac{\partial}{\partial y}Q = 0 \\ \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0. \end{cases}$$

We recall that in this α model, for $\alpha > 0$, u, v represent the ensemble average components of the velocity field.

Consider now the case of a two-dimensional steady incompressible viscous flow near the semi-infinite plate $\{(x, y) : x \geq 0, y = 0\}$. The following assumptions are made in this case:

- $\alpha = \sqrt{x}\beta$

- Zero pressure gradient, i.e. $\frac{\partial}{\partial x}Q = 0$.

In addition, we will study the solutions (u_∞, v_∞) of (4), which on some adequate interval $x_1 \le x \le x_2$ are of the form $u_\infty = f(\xi)$, $v_\infty = x^{-1/2}g(\xi)$, where $\xi = y/\sqrt{x}$.

Let $h(\xi) = \int_0^\xi f(\eta)d\eta$. Then $g = \frac{1}{2}\xi h' - \frac{1}{2}h$, and we have the following equation for h:

$$h''' + \frac{1}{2}hh'' - \beta^2\left(h''''' + \frac{1}{2}hh''''\right) = 0. \tag{5}$$

The boundary conditions are $h(0) = h'(0) = 0$, $h''(0) > 0$, and $h'(\xi) \to 1$ as $\xi \to \infty$. Note that if $h(\xi)$ is a solution of (5), then $\hat{h}(x) := \beta h(\beta x)$ is a solution of

$$h''' + \frac{1}{2}hh'' - \left(h''''' + \frac{1}{2}hh''''\right) = 0. \tag{6}$$

This equation can be also written as $m''' + \frac{1}{2}hm'' = 0$, where $m = h - h''$. Our main theoretical result is the following:

Theorem 1 *Given any $a > 0$, b, there exists $c(a, b)$ such that $h'(\xi) \to \gamma \ge 0$ as $\xi \to \infty$, where h is a solution of (6) with $h(0) = h'(0) = 0$, $h''(0) = a$, $h'''(0) = b$, and $h''''(0) = c$.*

For the proof, which is too long to be sketched here, see [7].

4 Comparison with Experimental Data

It is common to use

$$y^+ = \frac{u_\tau y}{\nu}, \quad u^+ = \frac{u}{u_\tau}$$

in the turbulent boundary layer, where $u_\tau = \sqrt{\nu \left.\frac{\partial u}{\partial y}\right|_{y=0}}$. Fix x on the horizontal axis and denote $l_* = \frac{\nu}{u_e}$, $R_x = \frac{x}{l_*}$. According to the derivation of (5) that was done in the section 3,

$$u = u_e h'\left(\frac{y}{\sqrt{l_* x}}\right)$$

represents a horizontal component of the averaged velocity for some h satisfying (5) with $h(0) = h'(0) = 0$, $h''(0) = a > 0$, $h'''(0) = b$, and $h'(\xi) \to 1$ as $\xi \to \infty$.

Such h is a rescaled solution of (6) provided by Theorem 1 in the generic case $\gamma \neq 0$. More precisely, $h(\xi) = \frac{1}{\beta}\hat{h}\left(\frac{\xi}{\beta}\right)$, where $\hat{h}$ satisfies (6) and $\beta^2 = \lim_{\xi\to\infty} \hat{h}'(\xi)$.

In terms of $\hat{h}$ we have

$$u^+ = \frac{R_x^{1/4}}{\sqrt{a}\beta^2}\hat{h}'\left(\frac{y^+}{\sqrt{a}\beta R_x^{1/4}}\right).$$

Given c_f, a skin-friction coefficient and R_θ, a Reynolds number based on momentum thickness we find a, b, and R_x so that the following conditions hold:

(1) $c_f = 2/\left(\inf_y u^+\right)^2$.
(2) $R_\theta = \frac{1}{\nu}\int_0^\infty u\left(1 - \frac{u}{u_e}\right)dy$.
(3) Von Karman log law only for the middle inflection point in logarithmic coordinates.

A family of curves $\{u\}_{c_f,R_\theta}$ was compared with experimental data of Rolls-Royce applied science laboratory, ERCOFTAC t3b test case (see Fig. 1). Comparison shows that the case $a + \beta b > 0$ corresponds to a laminar region of a boundary layer, $a + \beta b < 0$ corresponds to a turbulent region of a boundary layer. The case $a + \beta b = 0$ corresponds to a transition point.

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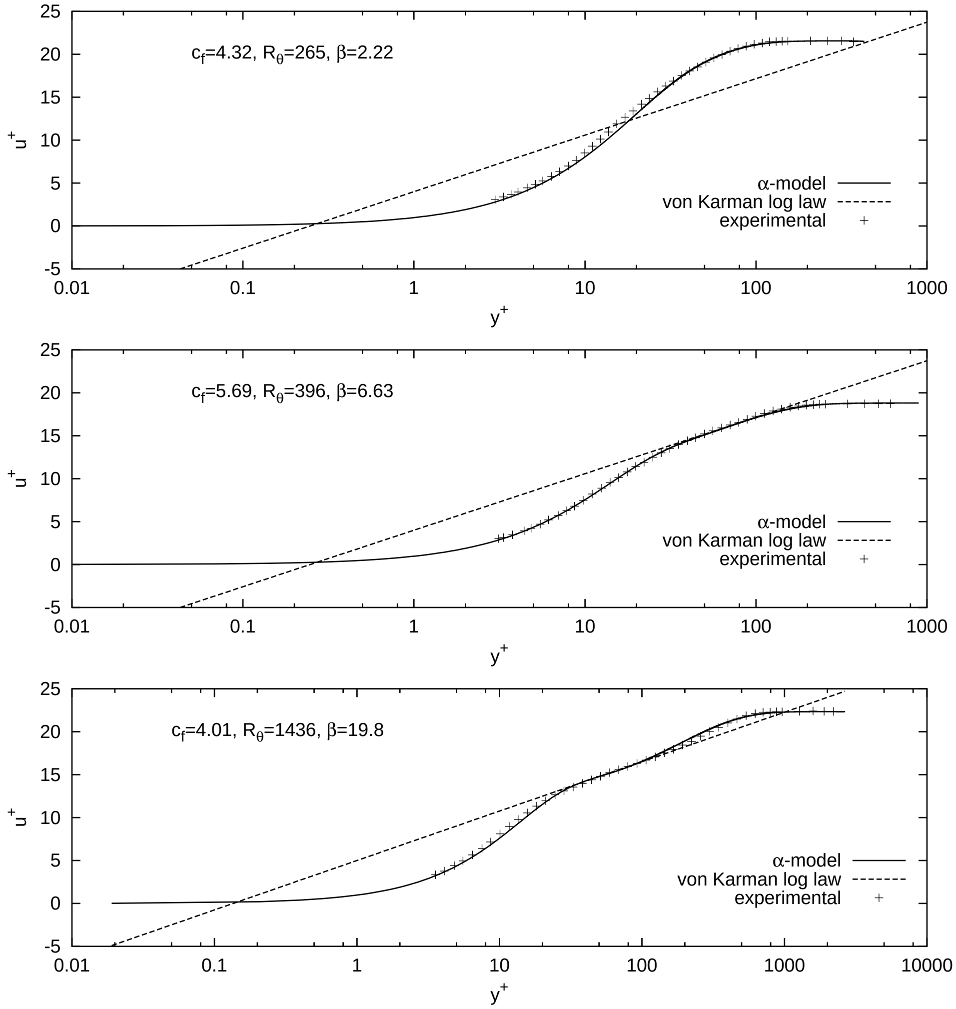

Fig. 1.

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