MATH 507 HOMEWORK 1 SOLUTIONS

(1) Show that if T is κ-categorical then for any $\lambda < \kappa$, T has no (κ, λ)-model.

Solution: Suppose not and that $M \models T$ is a (κ, λ) model of T. Let $\phi(\overline{x})$ be such that $|\phi(\overline{x})| = \lambda$. We show that there is $N \models T$ so that $|N| = \kappa$ and $|\phi(N)| = \kappa$. Thus $N \ncong M$ violating κ-categoricity. Let n be the arity of ϕ. To build N let us expand the language of T by adding new constants $c_i^j : i \in \kappa, j \in n\}$. and consider the theory T^* given by:

$$T \cup \{c_i^j \neq c_k^l : (i,j) \neq (k,l) \in \kappa \times n\} \cup \{\phi(c_i^1, \ldots c_i^n) : i \in \kappa\}.$$

It follows easily by compactness that T^* is consistent. Any $N \models T^*$ with $|N| = \kappa$ is as desired.

(2) Let $\mathfrak{R} = \langle \mathbb{R}, +, \cdot, < \rangle$ and let $T = Th(\mathfrak{R})$. Show that T has no Vaughtian pairs.

Solution: Suppose not and that $M \prec N$ is a Vaughtian pair witnessed by $\phi(x_1, \ldots x_n)$. It follows easily that if $n \neq 1$ then for some $1 \leq i \leq n$ $\exists x_1 \ldots \exists x_{i-1} \exists x_{i+1} \ldots \exists x_n \phi(\overline{x})$ will also produce a Vaughtian pair between M and N so without loss of generality $|\overline{x}| = 1$. Also without loss of generality we may reduce to the case that $\phi(x)$ defines an interval. We may also reduce to the case that $\phi(x)$ is a bounded interval in M and thus we have that $\phi(M) = \{x \in M : a < x < b\}$ for some $a, b \in M$. Let $n \in N \setminus M$. If either $n > m$ for all $m \in M$ or $n < m$ for all $m \in M$, replace n by n^{-1} and notice that $n^{-1} \in N \setminus M$. Hence without loss of generality we may assume that $n \in (c, d)$ for some $c, d \in M$. Now consider the function:

$$f(x) = \frac{d-c}{b-a}(x-a) + c.$$

f(x) is an M-definable bijection from (c, d) to (a, b). Thus $f(n) \in M$. But then since f is M-definable $f^{-1}(f(n)) \in M$ a contradiction.

(3) Show that if $M_0 \preceq M_1 \preceq \cdots \preceq M_i \preceq \ldots$ is an elementary chain for $i \in \kappa^+$ so that each M_i is κ^+-homogeneous then $\bigcup_{i \in \kappa} M_i$ is κ^+-homogeneous.

Solution: Let $M = \bigcup_{i\in\kappa} M_i$. Pick $A \subseteq M$ so that $|A| \leq \kappa$, $f : A \to M$ an elementary map, and $a \in M$ we need to show that we can extend f to f^* so that a is in the domain of f^*. Since $|A| \leq \kappa$, we may find $\alpha \in \kappa^+$ so that $A \subseteq M_\alpha$. To see this, suppose not. Let a_β with $\beta \in |A|$ be an enumeration of A. For each β let $g(\beta) \in \kappa^+$ be the least $\alpha \in \kappa^+$ so that $a_\beta \in M_\alpha$. Thus by assumption we have that $\kappa^+ = \bigcup_{\beta\in|A|} g(\beta)$, but this yields that κ^+ is the union of fewer than κ^+ sets each of size less than or equal to κ^+ which is impossible. Similarly we may find α so that $f(A) \subseteq M_\alpha$. Thus overall we may find an $\alpha \in \kappa^+$ so that $A \cup f(A) \cup \{a\} \subseteq M_\alpha$. Also since $M_\alpha \preceq M$, $f : A \to M_\alpha$ is elementary. Because M_α is κ^+-homogeneous we may find $f^* : A \cup \{a\} \to M_\alpha$ elementary extending f. But once again since $M_\alpha \preceq M$, f^* is an elementary map from $A \cup \{a\}$ to M and thus M is κ homogeneous.

(4) Give an example of a complete non-ω-stable theory which has a (κ, λ)-model for all infinite cardinals $\kappa > \lambda$.

Solution: Let $\mathcal{L}$ be the language consisting of a single binary relation $<$ and two constants c_0, c_1. Let T be dense linear order without endpoints together with the axiom $\{c_0 < c_1\}$. A simple back and forth argument show that T is ω-categorical and thus complete. To see that T is not ω-stable let $M = \langle \mathbb{Q}, <, 0, 1\rangle$ (we interpret c_0 as 0 and c_1 and 1). Notice that for every $r \in \mathbb{R} \setminus \mathbb{Q}$ if we let $p_r = \{x < q : r < q\} \cup \{x > q : r > q\}$ then the p_r are an uncountable collection of pairwise inconsistent types over M and hence that M is not ω-stable. Fix κ and λ. Let N_0 be a model of T of size λ and let N_1 be a dense linear order of size κ. Let $N = N_0^\frown N_1$ (the ordering consisting of N_0 followed by N_1), where the constants are interpreted as they were in N_0. Then $N \models T$ and $|N| = \kappa$ but $\{x \in N : c_0 < x < c_1\}$ has size λ.