MATH 507 HOMEWORK 2 HINT

For problem 2 it is useful to know the following:

Fact 1. *$T = Th((\mathbb{Z}4\mathbb{Z})^\omega)$ has quantifier elimination.*

Proof. (Sketch) First notice that in T any term is equivalent to one of the form

$$n_1y_1 + n_2y_2 + \cdots + n_my + m$$

where $n_i \in \{1, 2, 3\}$. Also any atomic formula is of the form $t(\overline{x}) = 0$ where t is a term. To establish quantifier elimination it suffices to show that a formula of the form

$$\exists x(A_1(x\overline{y}) \wedge \cdots \wedge A_m(x\overline{y}))$$

is equivalent to a quantifier free formula where the A_i are atomic or negated atomic. Notice that if one of the A_i is of the form $x + t(\overline{y}) = 0$ then we are done since this is equivalent to $x = 3t(\overline{y})$. Similarly if one of the A_i is of the form $3x+t(\overline{y}) = 0$ we are done since this is equivalent to $x = t(\overline{y})$. Next suppose that we have A_i and A_j distinct so that A_i is $2x + t(\overline{y}) = 0$ and A_j is $2x + s(\overline{y}) = 0$ then $A_i \wedge A_j$ is equivalent to $A_i \wedge s(\overline{y}) = t(\overline{y})$ so without loss of generality we may assume there is only one such conjunct. Thus we have reduced ourselves to a formulas of the form:

$$\exists x(2x + t(x) = 0 \wedge \bigwedge A_i) \text{ or } \exists x(\bigwedge A_i)$$

where the A_i are negated atomic. It is easy to see that in the second case the formula is equivalent to a quantifier free formula. In the first case notice that we may assume that none of the A_i are of the form $2x + s(\overline{y}) = 0$. Given this it is once again easy to see that the formula is equivalent to a quantifier free formula. □