THEORIES WITHOUT THE INTERMEDIATE VALUE PROPERTY AND WEAK O-MINIMALITY

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1. Introduction

In this short note we consider to what degree the notion of weak o-minimality relative to the class of ordered theories without the intermediate value property can be thought of as behaving analogously to the role of o-minimality in the class of ordered theories with the intermediate value property. For the problems we consider our answers are negative. Recall the definition of weak o-minimality:

Definition 1. (See for example [8] and the references therein.) A structure $(M, <)$ with a symbol $<$ for a dense linear order is called *weakly o-minimal* if any definable $X \subseteq M$ is a finite union of convex sets. A theory T is weakly o-minimal if all of its models are.

Also recall the definition of the intermediate value property (for a discussion of this see [10]):

Definition 2. A structure $(M, <)$ as in the previous definition is said to have the *intermediate value property (IVP)* if there are no open definable subsets $U, V \subseteq M$ such that $U \cup V = M$ and $U \cap V = \emptyset$

Recall that any o-minimal theory has the IVP (see [5]). Also notice that if M is weakly o-minimal but not o-minimal then the IVP must fail for M. Recall that in analogue to o-minimal theories weakly o-minimal structures or theories satisfy many desirable properties (see [8]). Hence we are led to our initial question, to what extent is weak o-minimality an appropriate analogue for o-minimality in the class of theories without the IVP?
In the ensuing two sections we consider two questions. First we consider a topological property, namely whether the projection of a closed bounded set in a weakly o-minimal theory is closed. Our work shows that the answer to this question is negative, but in specific cases the desired property may still hold. Secondly we focus on results concerning theories expanding divisible ordered abelian groups satisfying the intermediate value property which allow one to conclude that under various condition the theory is o-minimal or "close" to o-minimal. We ask whether we may remove the assumption of the intermediate value property if we weaken the conclusion to the

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analogous statement with “weakly o-minimal” in place of “o-minimal” . Our result is negative and the final section is devoted to counterexamples.

2. Projections of Closed Bounded Sets

As mentioned in the introduction if T is o-minimal, $M \models T$, $X \subseteq M^n$ is closed bounded and definable, and π is a projection function from M^n then $\pi(X)$ is closed. We begin by asking to what degree this holds for weakly o-minimal structures. We begin with a general result:

Proposition 1. *Let $\mathcal{M} = \langle M, +, <, ...\rangle$ be an archimedean model of the theory of divisible ordered abelian groups. Then the following are equivalent:*

(1) *$\mathcal{M}$ has the IVP.*

(2) *If X is a closed bounded $\mathcal{M}$-definable set and π is any projection then $\pi(X)$ is closed.*

Proof. (1) implies (2) was proved in greater generality by C. Miller in [10].

For (2) implies (1) we establish the contrapositive. So suppose that the IVP fails for M. Let U and V be definable open subsets of M such that $U \cap V = \emptyset$ and $U \cup V = M$. For any $c \in U$ let U_c be the convex component of U containing c, do the same for V (note that these are definable). Since U and V are disjoint we may assume without loss of generality that there is c in U such that U_c is bounded above. Since U and V are open there is no least upper bound and no greatest lower bound for U_c, in particular U_c is closed. Fix d some element of M larger than 0 such that $c - d \in U_c$ and consider the following definable subset of M^2 :

$$X := \{(x, y) \in M^2 : x \in [0, d],\ c - d \leq y,\ y \in U_c,\ \text{and } y + x \notin U_c\}.$$

Let $\pi(X)$ be the projection of X onto the first coordinate. Consider the following facts about X and $\pi(X)$:

(1) X is closed.

(2) $0 \notin \pi(X)$.

(3) $(0, d] \subseteq \pi(X)$.

The first two are clear from the definition of X. For the third fact fix $\epsilon \in (0, d]$, if $\epsilon \notin \pi(X)$ then for any $e \in U_c$ we must have that $e + \epsilon \in U_c$, which is impossible since M is archimedean. Hence we get that $\pi(x)$ is $(0, d]$ and thus not closed.

□

For the weak o-minimality of the following example see [1].

Example 1. *If $M = \langle \mathbb{R}^{alg}, +, \cdot, <, (-\pi, \pi)\rangle$, the real algebraics with a predicate for the interval between $-\pi$ and π, then there is a closed bounded definable set $X \subseteq M^2$ whose projection is not closed.*

We also show that in Proposition 1 the fact that we assume that the structure is that of an ordered group is in essence necessary. Of course we can not make sense of the term "Archimedean" in this case, but we give an example with universe $\mathbb{Q}$.
We need a simple topological fact:

Fact 1. *Let X be a topological space. If $A \subseteq X$, $U \subseteq X$, and $A \cap U$ is closed then $A \cap U = \overline{A} \cap U$.*

Proposition 2. *$\mathcal{Q} = \langle \mathbb{Q}, <, P, 0 \rangle$, the ordered rationals with a predicate, P, for the interval between $-\pi$ and π, has the property that projections of closed bounded definable sets are closed.*

Proof. This is elementary so we provide a sketch. First notice that $Th(\mathcal{Q})$ eliminates quantifiers. (Given a formula of the form $\exists x \phi(x, \overline{y})$ with ϕ quantifier free, one can easily write out an equivalent quantifier free formula.) Now let $\psi(x_1, .., x_n, a_{n+1}, ..., a_{n+m})$ be any complete quantifier free formula with $a_{n+1}, ..., a_{n+m}$ from M (by completeness we mean it completely determines the order type of the set

$$\{x_1, ..., x_n, a_{n+1}, ..., a_{n+m}, 0\}$$

and also whether $t \in (-\pi, \pi)$ for each term t appearing). Hence it has the form:

$$t_0 \Box_0 t_1 \Box_1 ... \Box_l t_{l+1} \wedge \bigwedge_{i \in J} Pt_i \wedge \bigwedge_{i \notin J} \neg Pt_i$$

where $t_1, ..., t_l$ lists all the terms appearing in $\psi(\overline{x}, \overline{a})$, J is some subset of $\{1, ..., l\}$, and $\Box_i \in \{<, =\}$. Suppose that the set thus defined is closed. Given any closed set defined in this way, by Fact 1 it is also defined by the formula of the same form with the symbol $<$ replaced by $\leq$, noting that both P and $\neg P$ define open sets. Given this, verifying that the projection of the closure of such a set is closed is elementary using the quantifier elimination. Since any definable set is a finite union of sets defined by complete quantifier free formulas we are done.

□

Note that by the main result in [1] the above structure is weakly o-minimal.
We do the same for the assumption of M being Archimedean in Proposition 1. We begin by pointing out that in Proposition 1 in order to prove that M without the IVP has a closed bounded definable set whose projection is not closed we can replace the archimedean assumption with:

> There is a pair of open convex sets U and V such that $U \cup V = M$, for all x, y if $x \in U$ and $y \in V$ then $x < y$, and there is no $\epsilon > 0$ such that for all $x \in U$ we also have that $x + \epsilon \in U$.

In the terminology of [8], this condition is met if M is of *non-valuational type*.
We show that this is necessary:

Proposition 3. *Let $\mathcal{R} = \langle \mathbb{R} \times \mathbb{Q}, <, +, 0, U, \lambda\rangle_{\lambda \in \mathbb{Q}\setminus\{0\}}$ where $<$ is lexicographic ordering, $+$ is componentwise addition, $U := \{(x, y) \in \mathbb{R} \times \mathbb{Q} : x = 0\}$, and λ is multiplication by the rational number λ. Then in $\mathcal{R}$, if X is a closed bounded set and π is any projection, $\pi(X)$ is closed.*

Proof. The proof of this fact is very much like the previous proposition. We first establish that the theory eliminates quantifiers. After some simple manipulation and omitting easy cases this reduces to two main cases. First we eliminate the existential quantifier from a formula of the form:

$$\phi := \exists y(t_0 < x < t_1 \wedge \bigwedge_{j=1}^{M} \neg U(\lambda_j y + r_j))$$

where t_0, t_1, r_i are all terms in which y does not appear. This formula is equivalent to the conjunction of

$$U(t_0 - t_1) \to t_0 < t_1 \wedge \bigwedge_{j=i}^{M} \neg U(\lambda_j t_0 + r_j)$$

and

$$\neg U(t_0 - t_1) \to t_0 < t_1.$$

The equivalence of ϕ and this conjunction follows from the following two facts:

(1) $\mathcal{R} \models \forall x \forall y \forall z (U(x - y) \to (U(\lambda x + z) \leftrightarrow (U\lambda y + z))$ for all λ.

(2) $\mathcal{R} \models \forall x \forall y \forall z_1 ... \forall z_n ((\neg U(x - y) \wedge x < y) \to \exists w (x < w < y \bigwedge_{i=1}^{n} \neg U \lambda_i w + z_i))$ for all n and $\lambda_1 ... \lambda_n$.

The first fact is immediate, the second follows since the set U interpreted in $\mathcal{R}$ is countable while if $a, b \in \mathcal{R}$ are such that $\mathcal{R} \models a < b \wedge \neg U(a - b)$ then the interval (a, b) in $\mathcal{R}$ is uncountable.

In the second case we must eliminate the quantifier from:

$$\exists y(t_0 < y < t_1 \wedge \bigwedge_{i=1}^{N} U(\lambda_i y + s_i) \wedge \bigwedge_{j=1}^{M} \neg U(\lambda_j y + r_j)$$

where the t's, r's, and s's are terms in which y does not appear. This is equivalent to:

$$\exists y(t_0 < y < t_1 \wedge U\lambda_1 y + s_1 \wedge \bigwedge_{i=2}^{N} U(\lambda_1^{-1} s_1 - \lambda_i^{-1} s_i) \wedge \bigwedge_{j=1}^{M} \neg U(\lambda_1^{-1} s_1 - \lambda_j^{-1} r_j)).$$

This follows since:

$$\mathcal{R} \models \forall x \forall y \forall z ((U(\lambda_1 x + z) \wedge U(\lambda_2 x + y)) \leftrightarrow (U(\lambda_1 x + z) \wedge U(\lambda_1^{-1} z - \lambda_2^{-1} y)))$$

So we are reduced to considering:

$$\exists y(t_0 < y < t_1 \wedge U(\lambda_1 y + s_1)).$$

This is readily seen to be equivalent to:

$$t_0 < t_1 \wedge (U(\lambda_1 t_0 + r_1) \vee U(\lambda_1 t_1 + r_1) \vee \\ (\lambda_1 t_0 + r_1 < 0 < \lambda_1 t_1 + r_1) \vee (\lambda_1 t_1 + r_1 < 0 < \lambda_1 t_0 + r_1)).$$

Hence we have quantifier elimination. Verifying that the projection of a closed set is closed is much like in the previous example using the explicit quantifier elimination given above.

□

Note that once again by [1] the above structure is weakly o-minimal.
We end this section by pointing out that our results contrast oddly with results in [8]. Namely in this paper the authors show that if M is a weakly o-minimal structure of non-valuational type then it should be considered to behave very similarly to an o-minimal structure. Here we show that having non-valuational type is exactly a condition that allows us to verify that the projection of a closed bounded definable set is not closed, in direct contrast to the o-minimal case. On the other hand we show that there are structures of valuational type where the desired property of projections does hold.

3. A Counterexample

In this section we wish to consider whether there are weakly o-minimal analogues of two results of Miller and Steinhorn. We begin with a definition:

Definition 3. ([12]) For a structure $(M, <)$ the *open core* of M (denoted M^o) is the reduct of M generated by all open definable subsets of M^n. We say that a theory T is of *o-minimal open core* if for any $M \models T$, M^o is o-minimal, the definition of *weakly o-minimal open core* is analogous.

The two results we with to consider are:

Theorem 1. *(Miller and Steinhorn in* [13]*)*

(1) *If M is a model of some expansion of the theory of divisible ordered abelian groups such that any definable $X \subseteq M$ is constructible and there is no $X \subseteq M$ which is infinite and discrete, then M is o-minimal.*
(2) *If T is some expansion of the theory of divisible ordered abelian groups which satisfies uniform finiteness and the IVP then T is of o-minimal open core.*

For definitions and facts regarding uniform finiteness, constructibility see [11].
The goal of this section is to show that the analogues of the above statements with the assumption of the IVP dropped and "o-minimal" replaced by "weakly o-minimal"

are false. We construct a single example to refute both of the above statements. The example shows that even if we strengthen the conditions in (2) above to include the exchange property for definable closure (which in fact implies uniform finiteness, see [4] for facts on the exchange property) the conclusion does not follow. Also our example will be a model of the theory of real closed fields, so assuming that we work with expansions of the theory of real closed fields rather than of divisible ordered abelian groups still does not yield a true statement.

We point out that the motivation for our example comes from [2], where the idea of adding a new random set to a structure is introduced. In our case we desire, rather than adding a truly random set, to add a new random open set. This should be made clear in the ensuing construction.

Let T_{RCF} be the theory of real closed fields formulated in language $\mathcal{L}_{df}$ where every definable function is given by a term in the language, so that T_{RCF} eliminates quantifiers and is universally axiomatizable. Fix $\mathcal{R}$ a sufficiently saturated model of T_{RCF}. Let V be a non-trivial convex subring of $\mathcal{R}$, and let $\mathcal{L}_{conv}$ be $\mathcal{L}_{df}$ augmented with a unary predicate for V. Let $(\mathcal{R}, V)$ be $\mathcal{R}$ formulated in $\mathcal{L}_{conv}$ in the obvious way, so $(\mathcal{R}, V)$ is a real closed valued field. Then we have:

Theorem 2. *(Van den Dries and Lewenberg in* [6]*, Cherlin and Dickmann in* [3]*)* $T_{conv} = Th((\mathcal{R}, V))$ *eliminates quantifiers in* $\mathcal{L}_{conv}$ *and is weakly o-minimal.* T_{conv} *may be axiomatized by stating that* $\mathcal{R}$ *is a real closed field and that* V *is a non-trivial convex subring.*

We need a simple fact about T_{conv}:

Lemma 1. T_{conv} *satisfies the exchange property for definable closure.*

Proof. Fix M an ω-saturated model of T_{conv}. We show that if $a \in \mathrm{dcl}(\bar{b})$ in M then $a \in \mathrm{dcl}_{T_{RCF}}(\bar{b})$, i.e. in the restriction of M to $\mathcal{L}_{RCF}$. Then by the o-minimality of T_{RCF} we have the desired result (see [5]). Let $\phi(x, \bar{b})$ be an algebraic formula so that $M \models \phi(a, \bar{b})$. By quantifier elimination and the universal axiomatizability of T_{RCF} we reduce to the case where $\phi(x, \bar{b})$ is of the form:

$$x \in I \wedge \bigwedge_{i=1}^{N} V f_i(x) \wedge \bigwedge_{j=1}^{M} \neg V g_j(x).$$

Here I is an interval with endpoints given by T_{RCF}-definable functions applied to $\bar{b}$ and the f's and g's are $\bar{b}$-definable functions in T_{RCF}. Without loss of generality we may assume that a lies in the interior of I, else we are done. In particular we may assume I is open. By o-minimality of T_{RCF} we may assume that the functions f and g are continuous, and strictly increasing or decreasing on I. But then notice that any set defined in this way is open, hence if it is non-empty it is infinite. Thus I must be finite and $a \in \mathrm{dcl}_{T_{RCF}}(\bar{b})$. □

Following Mellor (see [9]) may also think of a T_{conv} structure formulated in a three-sorted language $\mathcal{L}_3$ with sorts:

(1) $(\mathcal{R}, +, -, \cdot, 0, 1, <)$
(2) $(\Gamma, +, -, 0, \infty, <)$
(3) $(k, +, -, \cdot, 0, 1, <)$

Where Γ is the value group and k is the residue field.
We also need to add functions:

(1) $v : \mathcal{R} \to \Gamma$
(2) $Res : \mathcal{R}^2 \to k$.

We let T_3 be the $\mathcal{L}_3$ theory stating:

(1) $\mathcal{R}$ is a real closed field.
(2) $\Gamma \setminus \infty$ is a divisible ordered abelian group.
(3) k is a real closed field.
(4) v is the valuation map.
(5) Res is the map such that $Res(x, y) = xy^{-1}$ if $v(x) \geq v(y)$ and 0 otherwise.

Any model M of T_{conv} may also considered as a model of T_3, we will denote the associated $\mathcal{L}_3$-structure M_3.

Theorem 3. *(Mellor in [9]) T_3 eliminates quantifiers in $\mathcal{L}_3$.*

Corollary 1. *([9]) If $M \models T_3$, the induced structure on the group sort is just the divisible abelian group structure, while that on the residue field sort is just the real closed field structure.*

Given $M \models T_3$ let M_{conv} be the structure whose underlying set is the first sort in M formulated in the language $\mathcal{L}_{conv}$. Note that $M_{conv} \models T_{conv}$.
We need a simple fact about T_3:

Lemma 2. *T_3 satisfies uniform finiteness.*

Proof. This is immediate from the fact that o-minimal and weakly o-minimal theories satisfy uniform finiteness. □

Given this we now may augment $\mathcal{L}_3$ with a new predicate G to form $\mathcal{L}_3^G$. Fix $\widehat{M} \models T_3$, sufficiently saturated, we augment $\widehat{M}$ to a $\mathcal{L}_3^G$ structure $\widehat{M}^G$ by adding a "random" subset to the second sort as in [2]. We let T_3^G be the theory of $\widehat{M}^G$.
Working in $\widehat{M}^G$, we let $\widehat{M}^G_{conv}$ be the first sort formulated in the language $\mathcal{L}^G_{conv} := \mathcal{L}_{conv} \cup \{H\}$ with a new unary predicate. In $\widehat{M}^G_{conv}$ set:

$$H(\widehat{M}^G_{conv}) := \{x : \widehat{M}^G \models Gv(x)\}.$$

Let $T^G_{conv} = Th(\widehat{M}^G_{conv})$.
As above given $M \models T^G_{conv}$ we let M_3 be the associated $\mathcal{L}_3^G$-structure, setting:

$$G(M_3) := \{x \in \Gamma : M \models Gy \text{ for any } y \text{ such that } v(y) = x\}$$

We note an easy fact:

Lemma 3. *If* $M \models T^G_{conv}$ *then* $M_3 \models T^G_3$.

Proof. If $M \models T^G_{conv}$ then $M \equiv \widehat{M}^G$ by definition. Also $M^{eq} \equiv \widehat{M}^{G\,eq}$. But then we see that $M_3 \equiv \widehat{M}$, and hence $M_3 \models T^G_3$. □

We can now list the facts about T^G_{conv} which, in our case, are the most significant.

Theorem 4. *Let* $M \models T^G_{conv}$ *then:*

(1) $H(M)$ *is open and has infinitely many components, so* M^o *is not weakly o-minimal.*
(2) *Any definable* $X \subseteq M$ *is constructible.*
(3) T^G_{conv} *satisfies the exchange property for dcl.*
(4) *There are no definable* $X \subseteq M$ *which are infinite and discrete.*
(5) T^G_{conv} *satisfies uniform finiteness.*

Proof. For (1) simply note that for any x in the value group of M_3 $v^{-1}(x)$ is open. By the genericity of G for any $x, y \in G$, $v^{-1}(x)$ and $v^{-1}(y)$ are disjoint sets.

(2) Let $X \subseteq M$ be any definable set, note that X is also a definable set in M_3, hence it suffices to show that any definable subset of the first sort in M_3 is constructible. By the quantifier elimination for T_3 and properties of the generic extension (see [2]) a definable subset of the first sort is of the form:

$$\phi(x) \wedge \bigwedge_{i=1}^{N} G_i f(x) \wedge \bigwedge_{j=1}^{M} \neg G g_j(x)$$

where $\phi(x)$ is a formula in $\mathcal{L}_3$ and the f's and g's are T_3-definable functions from the first sort to the value group. For constructibility it thus suffices to show that if f is a T_3-definable function from the first sort to the value group then $f^{-1}(G)$ is constructible (that $f^{-1}(\neg G)$ is constructible will be identical). Hence suppose we have a formula $\psi(x, y, \bar{z})$ such that x is a variable in the value group and that for parameters $\bar{c}$ and an element b in the first sort, $\psi(M_3, b, \bar{c})$ is finite. Without loss of generality we may assume this definable set has cardinality one. A simple argument using quantifier elimination yields that if $M_3 \models \psi(a, b, \bar{c})$ then there are finitely many definable functions $g_1, ..., g_n$ from $\Gamma^N \to \Gamma$ (for some N), and finitely many definable functions $h_1, ..., h_m$ from M to M^N so that for any x in the domain of f, $f(x) = g_i(v(h_j(x)))$ for some i, j. Thus it suffices to show that for any $h : M \to M^N$ definable and $g : \Gamma^N \to \Gamma$ definable the set $h^{-1}(v^{-1}(g^{-1}(G)))$ is constructible. But note that $v^{-1}(g^{-1}(G))$ is open, then by the weak o-minimality of T_{conv} and the T_3-definability of h, $h^{-1}(v^{-1}(g^{-1}(G)))$ is constructible.

(3) It suffices to show that if $a \in \mathrm{dcl}(\bar{b})$ in M then $a \in \mathrm{dcl}_{T_{conv}}(\bar{b})$, i.e. the definable closure relation in M is the same as that in its restriction to $\mathcal{L}_{conv}$. Exchange then follows from Lemma 2.

Suppose that $a \in \mathrm{dcl}(\bar{b})$ in M. Then working in M_3 we also have that $a \in \mathrm{dcl}(\bar{b})$ in M_3, which is most easily seen by realizing that M_3 is interdefinable with a finite set of sorts from M^{eq}. But by the genericity of the set G, $a \in \mathrm{dcl}_{T_3}(\bar{b})$, i.e. thinking of M_3 as an $\mathcal{L}_3$ structure. But then we see that $a \in \mathrm{dcl}_{T_{conv}}(\bar{b})$ in M thought of as an $\mathcal{L}_{conv}$ structure (once again since M_3 is a fragment of M^{eq}).

Assertion (4) follows immediately from (3) (see [4]).

Assertion (5) is an immediate consequence of (4).

□

To finish this section we briefly point out that a much simpler example of the above behaviour can be obtained if we do not demand an expansion of a real closed field. We consider

$$\mathcal{Q} = \langle \mathbb{Q}, +, <, 0, \lambda \rangle_{\lambda \in \mathbb{Q}}$$

in a language $\mathcal{L}$ where λ is multiplication by λ. Let G be a generic set for $\mathcal{Q}$ as described in [7]. Let $\mathcal{L}^H := \mathcal{L} \cup \{H\}$ where H is a new unary predicate. Now let $\mathcal{Q}^H$ be the structure:

$$\langle \mathbb{Q} \times \mathbb{Q}, +, <, H, \lambda \rangle_{\lambda \in \mathbb{Q}}$$

where $<$ is lexicographic ordering, $+$ is componentwise addition, λ is multiplication of both components by λ and

$$H := \{(x, y) \in \mathbb{Q} \times \mathbb{Q} : \mathcal{Q} \models Gx\}.$$

As in the previous example we are adding a "random" open set. Let T^H be the theory of $\mathcal{Q}^H$. We may show:

Fact 2. *T^H has quantifier elimination and satisfies (1)-(5) of the previous theorem.*

Finally we point out that the bounded PRC fields provide another class of examples of ordered structures with the exchange property (see [14]), although they do not appear to have the desired constructibility properties of the examples given here.

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