A Polyhedral Method to compute all Affine Solution Sets of Sparse Polynomial Systems*

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Abstract

To compute solutions of sparse polynomial systems efficiently we have to exploit the structure of their Newton polytopes. While the application of polyhedral methods naturally excludes solutions with zero components, an irreducible decomposition of a variety is typically understood in affine space, including also those components with zero coordinates. We present a polyhedral method to compute all affine solution sets of a polynomial system. The method enumerates all factors contributing to a generalized permanent. Toric solution sets are recovered as a special case of this enumeration. For sparse systems as adjacent 2-by-2 minors our methods scale much better than the techniques from numerical algebraic geometry.

Key words and phrases. affine set, irreducible decomposition, Newton polytope, monomial map, permanent, polyhedral method, Puiseux series, sparse polynomial.

1 Introduction

Given is $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, a polynomial system of N polynomials $\mathbf{f} = (f_1, f_2, \ldots, f_N)$ in n unknowns $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. We assume our polynomials are *sparse* and only few (relative to the degrees) monomials appear with nonzero coefficient. The structure of sparse polynomials in several variables is captured by their Newton polytopes. A *polyhedral method* exploits the structure of the Newton polytopes to efficiently compute the solutions of the polynomial system. For the problem of computing solution sets in the intersection of some coordinate planes, the direct application of a polyhedral method fails, because the Newton polytopes change drastically when selected variables become zero.

If every polynomial in the system has the same Newton polytope, then the volume of that Newton polytope bounds the number of isolated solutions with nonzero coordinates, as proven

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in [28]. This theorem was generalized in [6] and its constructive proof was implemented in [39]. A more general algorithm was given in [21]. The problem of counting the number of isolated solutions in affine space was first addressed in [31], see also [32], and [33]. Stable mixed volumes, introduced in [22], give an upper bound on the number of isolated solutions in affine space. Methods to compute stable mixed volumes efficiently were proposed in [14] and [15].

The complexity of counting the total number of affine solutions of a system of n binomials (two monomials with nonzero coefficients) in n variables was shown as #P-complete [10]. In [19] combinatorial conditions for the existence of positive dimensional solution sets are given, for use in a geometric resolution [16].

Finiteness results in celestial mechanics were proven with polyhedral methods in [18], [24]. Tropical algebraic geometry, see e.g.: [9], and in particular the fundamental theorem by [26], [30], provides inspiration for a polyhedral computation of all positive dimensional solution sets. In [38], [1, 2, 3], Puiseux series were proposed to develop solution sets of polynomial systems, starting at infinity. Coordinate transformations, similar to the ones in [2], were applied in a more general setting in [23]. For parametric binomial systems, an algorithm (using the Smith normal form) of polynomial complexity for the solutions with nonzero values of the variables was presented in [17].

One of the earliest descriptions of software to compute a primary decomposition of binomial ideals were published in [7] and [29]. Recent algebraic algorithms are in [27]. In relation to the general binomial primary decomposition of [13], our motivation stems from Puiseux series (over $\mathbb{C}$) and the ideals we obtain are radical.

Our first contribution is to formulate the search for affine solution sets as the enumeration of all factors that contribute to a generalized permanent. This enumeration extends directly to general sparse systems. Our second contribution is a polyhedral method to compute Puiseux series for all affine solution sets. Thirdly, prototypes for the proposed algorithms are implemented in PHCpack in [37]. We tested our methods and software on the family of adjacent 2-by-2 minors, a problem described in [12] and [20]. For such sparse systems, our method scales much better than the techniques of numerical algebraic geometry [4].

2 Monomial Parametrizations of Affine Solution Sets

Toric ideals are introduced with monomial maps in [34, Chapter 4]. In this section we define the representations of solution sets of binomial systems and give examples to illustrate the difference between the toric and the affine case.

Monomials $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ in n variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ are defined by exponents $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$ and abbreviated as $\mathbf{x}^{\mathbf{a}}$. Denote $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. A binomial system consists of equations $c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} - c_{\mathbf{b}} \mathbf{x}^{\mathbf{b}} = 0$, with $c_{\mathbf{a}}, c_{\mathbf{b}} \in \mathbb{C}^*$ and $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$. If we are interested in *toric solutions*, i.e.: $\mathbf{x} \in (\mathbb{C}^*)^n$, then we normalize the equation $c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} - c_{\mathbf{b}} \mathbf{x}^{\mathbf{b}} = 0$ into $\mathbf{x}^{\mathbf{a}-\mathbf{b}} = c_{\mathbf{b}}/c_{\mathbf{a}}$. After normalization, we write a binomial system as $\mathbf{x}^A = \mathbf{c}$, where the matrix A collects the exponent vectors and the coefficient vector $\mathbf{c}$ stores the coefficients. The null space of A gives exponent vectors for a unimodular coordinate transformation leading to a monomial parametrization of the solution set.

Definition 2.1. A *monomial parametrization* of a d-dimensional solution set in $\mathbb{C}^n$ is

$$x_k = c_k t_1^{v_{1,k}} t_2^{v_{2,k}} \cdots t_d^{v_{d,k}}, \quad c_k \in \mathbb{C}^*, v_{i,k} \in \mathbb{Z}, \text{ for } i = 1, 2, \ldots, d \text{ and } k = 1, 2, \ldots, n. \quad (1)$$

Substituting (1) into $\mathbf{x^a}$ we can write $\mathbf{x^a} = c_1^{a_1} c_2^{a_2} \cdots c_n^{a_n} t_1^{\langle \mathbf{a}, \mathbf{v}_1 \rangle} t_2^{\langle \mathbf{a}, \mathbf{v}_2 \rangle} \cdots t_d^{\langle \mathbf{a}, \mathbf{v}_d \rangle}$ where $\langle \mathbf{a}, \mathbf{v}_i \rangle = a_1 v_{i,1} + a_2 v_{i,2} + \cdots + a_n v_{i,n}$. Because the d vectors $\mathbf{v}_i$, $i = 1, 2, \ldots, d$, span a basis for the null space of the vectors $\mathbf{a} - \mathbf{b}$ of the binomial equations $c_\mathbf{a} \mathbf{x^a} - c_\mathbf{b} \mathbf{x^b} = 0$, we have d free parameters in $\mathbf{t} = (t_1, t_2, \ldots, t_d)$. Collecting the d vectors $\mathbf{v}_i$ into the columns of an n-by-d matrix V, and the coefficients c_k of (1) in the vector $\mathbf{c}_V$, we abbreviate a monomial parametrization in (1) as $\mathbf{x} = \mathbf{c}_V \mathbf{t}^V$.

Example 2.2. Because we want invertible coordinate transformations, we may need fractional powers.

$$\mathbf{f}(\mathbf{x}) = \begin{cases} x_1^{80} - x_2^{21} x_3^2 = 0 \\ x_1^{54} - x_2^{15} x_4^2 = 0 \end{cases} \qquad M = \begin{bmatrix} 5 & 21/22 & 0 & 0 \\ 18 & 80/22 & 0 & 0 \\ 11 & 0 & 1 & 0 \\ 0 & -33/22 & 0 & 1 \end{bmatrix} \qquad \begin{cases} x_1 = y_1^5 y_2^{21/22} \\ x_2 = y_1^{18} y_2^{80/22} \\ x_3 = y_1^{11} y_3 \\ x_4 = y_2^{-33/22} y_4 \end{cases} \quad (2)$$

$$A = \begin{bmatrix} -80 & 21 & 2 & 0 \\ -54 & 15 & 0 & 2 \end{bmatrix}$$

The first two columns of M span the null space of A. The denominator 22 is obtained from the pivot of the Hermite normal form of an integer vector that spans the null space of A. Dividing the columns of B by these pivots gives an extended matrix M with $\det(M) = 1$. We denote the coordinate transformation defined by M as $\mathbf{x} = \mathbf{y}^M$. Because M contains the null space of A, $\mathbf{f}(\mathbf{x} = \mathbf{y}^M) = \mathbf{0}$ contains only y_3 and y_4 (after clearing the common powers of y_1 and y_2). Solving $\mathbf{f}(\mathbf{y}) = \mathbf{0}$ gives the coefficients $c_3 = \pm 1$ and $c_4 = \pm 1$ we put in place for y_3 and y_4: $(x_1 = t_1^5 t_2^{21/22}, x_2 = t_1^{18} t_2^{80/22}, x_3 = (\pm 1) t_1^{11}, x_4 = (\pm 1) t_2^{-33/22})$, or equivalently: $(x_1 = s_1^5 s_2^{21}, x_2 = s_1^{18} s_2^{80}, x_3 = (\pm 1) s_1^{11}, x_4 = (\pm 1) s_2^{-33}, t_1 = s_1, t_2 = s_2^{22})$. By the auxiliary parameters s_1 and s_2, the parametrization has integer exponents.

For a correct determination of the degree of the solution set, we need unimodular monomial parametrizations. Using the abbreviated notation $\mathbf{x} = \mathbf{c}_V \mathbf{t}^V$, we extend Definition 2.1.

Definition 2.3. A *unimodular monomial parametrization* of a d-dimensional solution set in $\mathbb{C}^n$ is

$$(\mathbf{x}, \mathbf{t}) = (\mathbf{c}_V \mathbf{s}^V, \mathbf{s}^W) \quad \text{or} \quad (\mathbf{x} = \mathbf{c}_V \mathbf{s}^V, \mathbf{t} = \mathbf{s}^W), \quad (3)$$

where $V \in \mathbb{Z}^{n \times d}$ and $W \in \mathbb{Z}^{d \times d}$. The columns of V span the null space of the exponent vectors of the binomials and W is a diagonal matrix containing the denominators of the columns of V so that when VW^{-1} is extended with unit vectors into the square matrix M, $\det(M) = 1$ and $\mathbf{x} = \mathbf{y}^M$ is unimodular.

We assume that all our monomial parametrizations are unimodular and omit $\mathbf{s}$ when $W = I$.

An *affine solution set* is a component of a solution set contained in a subspace spanned by one or more coordinate hyperplanes. Some coordinates of an affine solution are zero, some are free, and others are linked to a toric solution of a subset of the original equations. We illustrate the distinction between variables in the next example and make this distinction precise in Definition 2.5, extending Definition 2.1 one last time.

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Example 2.4. An interesting class of examples are the adjacent minors (see [12, 20, 35]). Consider all adjacent 2-by-2 minors of a general 2-by-4 matrix X:

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix} \qquad \mathbf{f}(\mathbf{x}) = \begin{cases} x_{11}x_{22} - x_{21}x_{12} = 0 \\ x_{12}x_{23} - x_{22}x_{13} = 0 \\ x_{13}x_{24} - x_{23}x_{14} = 0 \end{cases} \tag{4}$$

which has a 5-dimensional toric solution $(x_{11} = t_1t_4t_5, x_{12} = t_2, x_{13} = t_3, x_{14} = t_5, x_{22} = t_2t_4^{-1}t_5^{-1}$, $x_{21} = t_1, x_{23} = t_3t_4^{-1}t_5^{-1}, x_{24} = t_4^{-1})$ of degree four and two affine solutions, each of degree two. Giving the variables in the third column of X the value zero reduces the original system to one equation. The variables x_{14} and x_{24} no longer occur in the remaining equations and are free. The other variables are interlinked. For each variable in the solution we explicitly indicate its type: $(x_{11} = t_1t_2t_3(\text{link}), x_{12} = t_3(\text{link}), x_{13} = 0(\text{zero}), x_{14} = t_4(\text{free}), x_{21} = t_2(\text{link}), x_{22} = t_1^{-1}(\text{link}), x_{23} = 0(\text{zero}), x_{24} = t_5(\text{free}))$. The other affine solution with $x_{12} = 0$ and $x_{22} = 0$ is obtained by symmetry.

Definition 2.5. An *affine monomial parametrization* of a solution set of a binomial system is a tuple associating to the variables one of the three types, *zero*, *free*, or *link*:

$$x_k = \begin{cases} 0 & zero \\ t_k & free \\ c_k\mathbf{t^v} & link \end{cases} \tag{5}$$

where $c_k \in \mathbb{C}^*$, $\mathbf{t} = (t_1, t_2, \ldots, t_d)$, d is the size of the set of parameters that control the link variables, $\mathbf{v} \in \mathbb{Z}^d$, and $\mathbf{t^v} = t_1^{v_1}t_2^{v_2}\cdots t_d^{v_d}$. In particular, the distinction between a free and a link variable is that the parameter t_k does not occur anywhere else in the monomial parametrization of the affine solution.

Proposition 2.6. *The degree of a solution component of dimension D of a binomial system given by an affine unimodular monomial parametrization equals the volume of the polytope spanned by the origin and the exponent vectors of all parameters. This polytope is described as follows. Let $D = d+e$, where d is the number of parameters that control the link variables and e is the number of free variables. Relabel the free variables so they come before the link variables. Then for every free variable k we have the kth standard basis vector $\mathbf{e}_k \in \mathbb{Z}^D$ and insert e zeros to each $\mathbf{v} \in \mathbb{Z}^d$.*

Proof. To compute the degree of a D-dimensional solution set of $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, we add D generic hyperplanes $L(\mathbf{x}) = \mathbf{0}$ and count the number of isolated solutions of $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ that satisfy $L(\mathbf{x}) = \mathbf{0}$. By the monomial parametrization, we can eliminate the original $\mathbf{x}$ variables, omit the original equations $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, and consider only the system $L(\mathbf{t}) = \mathbf{0}$. As the coefficients of the hyperplanes in L are generic, all equations have the same monomials and exponents: $L(\mathbf{t}) = \mathbf{0}$ has D equations in D unknowns. The theorem of [28] applies and the number of isolated solutions of $L(\mathbf{t}) = \mathbf{0}$ equals the volume of the Newton polytope spanned by the exponents of the polynomials in $L(\mathbf{t}) = \mathbf{0}$. □

Remark 2.7. For ease of notation, we assumed $W = I$ in Proposition 2.6. If $W \neq I$, then the volume of the polytope must be divided by $\det(W)$ to obtain the correct degree. For example, the solutions of Example 2.2 have degree $54 = 1188/22$. Proposition 2.6 is similar to [34, Theorem 4.16].

3 A Generalized Permanent

To enumerate all choices of variables to be set to zero, we use the matrix of exponents of the monomials to define a bipartite graph between monomials and variables. The incidence matrix of this bipartite graph is defined below.

Definition 3.1. Let $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ be a system. We collect all monomials $\mathbf{x^a}$ that occur in $\mathbf{f}$ along the rows of the matrix, yielding the *incidence matrix*

$$M_{\mathbf{f}}[\mathbf{x^a}, x_k] = \begin{cases} 1 & \text{if} \quad a_k > 0 \\ 0 & \text{if} \quad a_k = 0. \end{cases} \tag{6}$$

Variables which occur anywhere with a negative exponent are dropped.

Example 3.2. For all adjacent minors of a 2-by-3 matrix, the matrix linking monomials to variables is

$$M_{\mathbf{f}} = \begin{array}{c|cccccc} & x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} \\ \hline x_{11}x_{22} & 1 & 0 & 0 & 0 & 1 & 0 \\ x_{21}x_{12} & 0 & 1 & 0 & 1 & 0 & 0 \\ x_{12}x_{23} & 0 & 1 & 0 & 0 & 0 & 1 \\ x_{22}x_{13} & 0 & 0 & 1 & 0 & 1 & 0 \end{array}, \quad \text{for} \quad \mathbf{f} = \begin{cases} x_{11}x_{22} - x_{21}x_{12} = 0 \\ x_{12}x_{23} - x_{22}x_{13} = 0. \end{cases} \tag{7}$$

For this example, the rows of $M_{\mathbf{f}}$ equal the exponents of the monomials. We select x_{12} and x_{22} as variables to be set to zero, as overlapping columns x_{12} with x_{22} gives all ones.

Proposition 3.3. *Let S be a subset of variables such that for all $\mathbf{x^a}$ occurring in $\mathbf{f}(\mathbf{x}) = \mathbf{0}$: $M[\mathbf{x^a}, x_k] = 1$, for $x_k \in S$, then setting all $x_k \in S$ to zero makes all polynomials of $\mathbf{f}$ vanish.*

Proof. $M[\mathbf{x^a}, x_k] = 1$ means: $x_k = 0 \Rightarrow \mathbf{x^a} = 0$. If the selection of the variables in S is such that all monomials in the system have at least one variable appearing with positive power, then setting all variables in S to zero makes all monomials in the system vanish. □

Enumerating all subsets of variables so that $\mathbf{f}$ vanishes when all variables in a subset are set to zero is similar to a row expansion algorithm on $M_{\mathbf{f}}$ for a permanent, sketched in Algorithm 3.4.

Algorithm 3.4 (recursive subset enumeration via row expansion of permanent)**.**
    Input: $M_{\mathbf{f}}$ is the incidence matrix of $\mathbf{f}(\mathbf{x}) = \mathbf{0}$;
           index of the current row in $M_{\mathbf{f}}$; and S is the current selection of variables.
    Output: all S that make the entire $\mathbf{f}$ vanish.
    if $M[\mathbf{x^a}, x_k] = 1$ for some $x_k \in S$
      then print S if $\mathbf{x^a}$ is at the last row of $M_{\mathbf{f}}$ or else go to the next row
      else for all k: $M[\mathbf{x^a}, x_k] = 1$ do
          $S := S \cup \{x_k\}$
          print S if $\mathbf{x^a}$ is at the last row of $M_{\mathbf{f}}$ or else go to the next row
          $S := S \setminus \{x_k\}$

To limit the enumeration, every variable set to zero cuts the dimension of the solution set by one. If we have a threshold on the dimension of the solution set, then the enumeration stops

if the number of selected variables exceeds the threshold on the codimension. In the context of algebraic sets, a greedy enumeration should first search for the highest dimensional components and taking into account the frequencies of the variables occurring in each monomial, select the most frequently occurring variables first.

The above algorithm returns subsets of variables that make the entire binomial system vanish. For partial cancellation, note that skipping certain binomials means skipping pairs of rows in $M_{\mathbf{f}}$. The odd (respectively even) rows of $M_{\mathbf{f}}$ store the first (respectively second) monomial. Then the extra branch test in the enumeration proceeds as follows. If the current row in $M_{\mathbf{f}}$ is odd and if none of the selected variables occur in the current $\mathbf{x^a}$ and in $\mathbf{x^b}$ on the following row, then skip the row in one branch of the enumeration. Skipping one binomial equation implies that variables occurring with positive power in $\mathbf{x^a}$ and $\mathbf{x^b}$ should not be selected in the future. The skipped binomial equations define a toric solution for some variables in an affine monomial parametrization.

4 Membership Tests

Regardless of efficient greedy enumeration strategies, we still need a criterion to decide whether no member of a collection of affine monomial parametrizations is contained in another parametrization. We introduce the problem with an example.

Example 4.1. To illustrate the hierarchies of variables and monomials when skipping equations we consider the system taken from [20, Example 2.2]:

$$\mathbf{f}(\mathbf{x}) = \begin{cases} x_1 x_3^2 - x_2 x_6^2 = 0 \\ x_4 x_6^3 - x_1^3 x_5 = 0 \\ x_1 x_2 x_5 - x_4 x_6^2 = 0. \end{cases} \tag{8}$$

The system has two 3-dimensional toric solution sets: $(x_1 = t_1^2 t_2^2 t_3, x_2 = t_1^4 t_2^4 t_3, x_3 = \pm t_1 t_2 t_3, x_4 = t_1^6, x_5 = t_2^{-6}, x_6 = t_3)$, one 4-dimensional affine solution set: $(x_1 = 0, x_2 = t_1, x_3 = t_2, x_4 = t_3, x_5 = t_4, x_6 = 0)$, and three 3-dimensional affine solution sets: $(x_1 = 0, x_2 = 0, x_3 = t_1, x_4 = 0, x_5 = t_2, x_6 = t_3)$, $(x_1 = t_1, x_2 = t_2, x_3 = t_3, x_4 = 0, x_5 = 0, x_6 = 0)$, and $(x_1 = t_1 t_2^2 t_3^2, x_2 = t_1, x_3 = t_2^{-1}, x_4 = 0, x_5 = 0, x_6 = t_3)$. In the enumeration, after $x_1 = 0$ and $x_6 = 0$ have been found to completely set the system to zero, any additional sets of variables that include x_1 and x_6 should no longer be considered. Variables x_1 and x_6 occur most often in the system and if we order the variables along their frequency of occurrence, then $x_1 = 0$ and $x_6 = 0$ will be considered first, before all other pairs, and subsets containing x_1 and x_6.

Two equivalent (as defined below) monomial parametrizations describe the same solution set.

Definition 4.2. Consider two monomial parametrizations $\mathbf{c}_V \mathbf{t}^V$ and $\mathbf{c}_W \mathbf{t}^W$. If $\mathbf{c}_V = \mathbf{c}_W$ and the matrices V and W span the same linear space, then we say that the monomial parametrizations $\mathbf{c}_V \mathbf{t}^V$ and $\mathbf{c}_W \mathbf{t}^W$ are *equivalent*. Two affine monomial parametrizations are *equivalent* if the same variables are zero, the same variables are free, and moreover, their link variables have equivalent monomial parametrizations.

We have to be able to decide whether an affine monomial representation belongs to another (affine) monomial parametrization. We introduce this problem in the following example.

Example 4.3 (Example 4.1 continued). Consider (8). The enumeration generates $C_1 = (x_1 = t_1, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = t_2)$. But as it turns out, this component is a subset of $C_2 = (x_1 = t_1 t_2^2 t_3^2, x_2 = t_1, x_3 = t_2^{-1}, x_4 = 0, x_5 = 0, x_6 = t_3)$. This is not at all obvious from the given parametrization of C_2 because x_3 cannot become zero in C_2 because of the negative power of t_2. With some manipulations, we can find an equivalent parametrization for C_2: $(x_1 = t_1, x_2 = t_1 t_2^2, x_3 = t_2 t_3, x_4 = 0, x_5 = 0, x_6 = t_3)$ and then $t_2 = 0$ leads to C_1. A better way to consider whether $C_1 \subseteq C_2$ is to observe that C_1 is a monomial ideal, that is: the ideal $I(C_1)$ defined by all polynomials that vanish at C_1 is generated by $\langle x_2, x_3, x_4, x_5 \rangle$. We have $I(C_2) = \langle x_4, x_5, x_1 x_3^2 - x_2 x_6^2 \rangle$. Comparing $I(C_2)$ with $I(C_1)$, we observe that $x_1 x_3^2 - x_2 x_6^2 = (x_1 x_3) x_3 + (-x_6^2) x_2 \in I(C_1)$ and thus $I(C_2) \subseteq I(C_1)$, which implies $C_1 \subseteq C_2$. Verifying whether a polynomial belongs to a monomial ideal seems easier than finding an equivalent parametrization with positive powers at the right places.

Given an affine monomial parametrization C of a solution set $V(C)$ of a binomial system, the ideal $I(C)$ of all polynomials that vanish at $V(C)$ consists of monomials (defined by those variables that are set to zero) and binomial relations (defined by the power products in the parametrizations). The monomial parametrizations of the affine solution sets remove the multiplicities, e.g.: $(x - y)^2$ turns into $x - y$, and we therefore have that $V(I(C)) = C$, for any affine component C.

Algorithm 4.4 (defining equations via circuit enumeration).
  Input: C an affine monomial map of solutions $V(C)$.
  Output: $E(C)$ equations that define $V(C)$.

The implementation of Algorithm 4.4 considers all smallest affine dependencies of the exponents of the monomials. A smallest affine dependency between points is a circuit [8].

In the proposition below we formalize the ideal inclusion property according to our notations.

Proposition 4.5. *Let C_1 and C_2 be two affine monomial parametrizations for solutions sets of $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. Then $C_1 \subseteq C_2 \Leftrightarrow I(C_1) \supseteq I(C_2)$.*

Although the enumeration of all equations that define $V(C)$ has once again a combinatorial complexity, often only one particular equation solves the inclusion problem, illustrated next.

Example 4.6 (Example 4.1 continued). Could a toric component include the set defined by $x_4 = 0$ and $x_5 = 0$? To answer this question, it suffices to consider coordinates of the toric component that do not involve x_4 and x_5, for example: $x_1 = t_1^2 t_2^2 t_3, x_2 = t_1^4 t_2^4 t_3, x_6 = t_3$. The monomials in the parameters define the exponent matrix A and its null space defines a vanishing binomial:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 4 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \mathbf{0}, \quad x_1^2 - x_2 x_6 = 0, \quad x_1^2 - x_2 x_6 \notin \langle x_4, x_5 \rangle. \tag{9}$$

Therefore, the set with $x_4 = 0$ and $x_5 = 0$ does not belong to the set defined by $x_1^2 - x_2 x_6 = 0$.

Summarizing the properties of $M_{\mathbf{f}}$ and Proposition 4.5, we state that all irreducible components of the solution set of a binomial system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ are factors contributing to the generalized permanent of the incidence matrix $M_{\mathbf{f}}$. Moreover, the affine parametrizations of the components give enough equations to determine that every component reported by the enumeration does not belong to any other component.

5 Enumerating All Candidate Affine Solution Sets

To find all candidate affine solution sets, we sketch an extension of Algorithm 3.4.

Example 5.1 (Our running example). Consider [19, example 8]:

$$\begin{cases} x_1x_4 + x_1^2x_4^2 + x_1x_2x_3 + x_2x_3 = 0 \\ x_1x_2 + x_1x_2^2 + x_1x_3x_4 + x_3x_4 + x_3x_4^2 = 0 \\ x_1x_2x_4 + x_1x_3x_4 + x_2x_3 + x_2x_3x_4 = 0 \\ x_1 + x_1^2 + x_1x_2 + x_3^2 + x_3x_4 = 0 \end{cases} \tag{10}$$

where we have taken all coefficients to be equal to one. Notice that x_1 and x_3 appear in every monomial, so setting x_1 and x_3 to zero yields a 2-dimensional solution set.

If x_1 is set to zero, then also x_1^2 becomes zero, so in the incidence matrix we consider only those monomials which are not divided by any other monomial, as formalized in the next definition.

Definition 5.2. The supports of $\mathbf{f} = (f_1, f_2, \ldots, f_N)$ are $(A_1, A_2, \ldots, A_N)$: $f_i(\mathbf{x}) = \sum_{\mathbf{a} \in A_i} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$.

The *incidence matrix* for f_i is M_{f_i}: for all $\mathbf{a} \in A_i$ for which there is no $\mathbf{b} \in A_i \setminus \{\mathbf{a}\}$ such that $\mathbf{x}^{\mathbf{b}}$ divides $\mathbf{x}^{\mathbf{a}}$:

$$M_{f_i}[\mathbf{x}^{\mathbf{a}}, x_k] = \begin{cases} 1 & \text{if } a_k > 0 \\ 0 & \text{if } a_k = 0 \end{cases} \quad \text{and} \quad M_{\mathbf{f}} = \begin{bmatrix} M_{f_1} \mid M_{f_2} \mid \cdots \mid M_{f_N} \end{bmatrix}^T. \tag{11}$$

Example 5.3 (Example 5.1 continued). The incidence matrix for (10) is

$$M_{\mathbf{f}} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}^T. \tag{12}$$

Observe that the transposed matrix is displayed. The rows of $M_{\mathbf{f}}^T$ are indexed by the variables.

Running through the columns of $M_{\mathbf{f}}$ seems equivalent to enumerating all subsets of $\{x_1, x_2, \ldots, x_n\}$. Organizing the search along the rows of $M_{\mathbf{f}}$ allows for a greedy version, see Figure 1. For example, we could first set those variables to zero which appear most frequently in the monomials. Running Algorithm 3.4 through all equations, we obtain solutions that make all equations of $\mathbf{f}$ vanish.

6 A Polyhedral Method

Skipping a binomial equation, e.g.: $x_{11}x_{22} - x_{21}x_{12} \neq 0$ implies $x_{11} \neq 0$, $x_{22} \neq 0$, $x_{21} \neq 0$, and $x_{12} \neq 0$. For general polynomial equations, it suffices that *at least two* monomials remain. A purely combinatorial criterion is to consider all possible binomials to determine which variables should be nonzero.

$$M_{\mathbf{f}} = \begin{bmatrix} \textcircled{1} & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & \textcircled{1} & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}^{T}$$

Figure 1: Searching greedily, we first select $x_1 = 0$. Then we look for the first monomial that does not contain x_1 and we choose x_3 over x_2 because x_3 appears in more monomials.

In a polyhedral method we examine inner normals to determine initial form systems. For every skipped polynomial p, we consider all edges of its Newton polytope. For each edge e with inner normal cone V, let $\mathrm{in}_V(p)$ be the initial form of p: $\mathrm{in}_V(p)$ contains those terms $c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}}$ of p for which $\langle \mathbf{a}, \mathbf{v} \rangle$ is minimal for all $\mathbf{v}$ in the interior of the cone V. Instead of the pure combinatorial criterion from above, we now require that all variables occurring in $\mathrm{in}_V(p)$ should remain nonzero. The intersection of the inner normal cones of equations that are skipped determine the pretropism(s). Our polyhedral method to enumerate all candidate affine solution sets has input/output specification in Algorithm 6.1.

Algorithm 6.1 (input/output specification of polyhedral method)**.**
    Input: $M_{\mathbf{f}}$, the incidence matrix of $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. $E = (E_1, E_2, \dots, E_N)$,
        E_i is the set of all edges of the Newton polytope of f_i, $i = 1, 2, \dots, N$.
    Output: $S = \{\, (s, e) \mid n\text{-tuple } s : s_i = 0 \text{ if } x_i = 0,\ s_i = +1 \text{ if } x_i \in \mathbb{C},\ s_i = -1 \text{ if } x_i \neq 0;\text{ and}$
            $N\text{-tuple } e :\ e_i = \emptyset \text{ or } e_i \in E_i,\ \text{for } i = 1, 2, \dots, N \,\}.$

Skipping all equations and making all edge-edge combinations yields the refinement of normal cones for the tropical prevariety. Normal cone intersections prune superfluous combinations. Consider the processing of a tuple (s, e). If the dimension of the normal cone defined by $e_i \neq \emptyset$ equals D, then we have D parameters $t_1, t_2, \dots, t_D$. We have $D = \#\{\, s_i = -1 \mid (s, e) \in S \,\}$. For all $s_i = 1$, we have extra free variables $x_i = t_{D+j}$, for $j = 1, 2, \dots, \#\{\, s_i = +1 \mid (s, e) \in S \,\}$.

The specification of Algorithm 6.1 fits the description of the normal cone intersection algorithms to compute the tropical prevariety as defined by [9] and done by the software Gfan of [25]. Solutions to the initial form systems give the leading powers of Puiseux series expansions. Before we formalize the format of these expansions, we continue our running example.

Example 6.2 (Example 5.1 continued)**.** There are five cases that lead to affine solution sets:

1. Setting $x_1 = 0$ and $x_2 = 0$ leaves only $x_3 x_4^2 + x_3 x_4 = 0$ and $x_3^2 + x_3 x_4 = 0$. The solutions are the line $(x_1 = 0, x_2 = 0, x_3 = 0, x_4 = t_1)$ and $(0, 0, 1, -1)$.

2. Setting $x_2 = 0$ and $x_3 = 0$ leaves only $x_1^2 x_4^2 + x_1 x_4 = 0$ and $x_1^2 + x_1 = 0$. The solutions are the line $(x_1 = 0, x_2 = 0, x_3 = 0, x_4 = t)$, $(-1, 0, 0, 0)$, and $(-1, 0, 0, 1)$.

3. Setting $x_2 = 0$ and $x_4 = 0$ leaves only $x_1^2 + x_3^2 + x_1 = 0$. A Puiseux expansion for the solution starts as $(x_1 = t^2(-1 + O(t^2)), x_2 = 0, x_3 = t(-1 + O(t^2)), x_4 = 0)$.

4. Setting $x_3 = 0$ and $x_4 = 0$ leaves only $x_1 x_2^2 + x_1 x_2 = 0$ and $x_1^2 + x_1 x_2 + x_1$. The solutions are the line $(x_1 = 0, x_2 = t, x_3 = 0, x_4 = 0)$ and $(-1, 0, 0, 0)$.

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5. Setting $x_2 = 0$, $x_3 = 0$ and $x_4 = 0$ leaves $x_1^2 + x_1 = 0$, with solutions $(-1, 0, 0, 0)$ and $(0, 0, 0, 0)$.

Stable mixed volumes will also lead to all isolated solutions in affine space.

Proposition 6.3. *Assume* $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ *has an affine solution set with* ℓ *link variables,* m *free variables, and the remaining* $n - \ell - m$ *variables are set to zero. Ordering variables so the link variables appear first, followed by the free and then the zero variables, we partition* $\mathbf{x}$ *as*

$$\mathbf{x} = (x_1, x_2, \ldots, x_\ell, x_{\ell+1}, x_{\ell+2}, \ldots, x_{\ell+m}, x_{\ell+m+1}, x_{\ell+m+2}, \ldots, x_n). \tag{13}$$

Ordering the parameters so the first D *parameters* $t_1, t_2, \ldots, t_D$ *occur in the link variables:*

$$\mathbf{x} = \begin{cases} x_k = c_k \prod_{j=1}^{D} t_j^{v_{k,j}}(1 + O(\mathbf{t})) & k = 1, 2, \ldots, \ell \\ x_{\ell+k} = t_{D+k} & k = 1, 2, \ldots, m \\ x_{\ell+m+k} = 0 & k = 1, 2, \ldots, n - \ell - m \end{cases} \tag{14}$$

with coefficients $c_k \in \mathbb{C}^*$, $k = 1, 2, \ldots, \ell$ *and where the vectors* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_D \in \mathbb{Z}^k$ *span a D-dimensional cone V. Take any* $\mathbf{v} \in V$ *and denote* $\mathbf{w} = (\mathbf{v}, \mathbf{0}, \infty)$, *where* $\mathbf{0}$ *is a vector of m zeros and* ∞ *a vector of $n - \ell - m$ infinite numbers. Then, for* $\mathbf{z} = (x_1, x_2, \ldots, x_\ell, t_{D+1}, t_{D+2}, \ldots, t_{D+m}, 0, 0, \ldots, 0)$:

$$\mathrm{in}_{\mathbf{v}}(\mathbf{f}(\mathbf{z})) = (\mathrm{in}_{\mathbf{w}} \mathbf{f})(\mathbf{z}), \text{ for all values } t_{D+k} \in \mathbb{C}, k = 1, 2, \ldots, m. \tag{15}$$

Proof. To prove (15) we consider two cases. When the last $n - \ell - m$ variables are zero, either an equation vanishes entirely or some monomials remain. For monomials $\mathbf{x}^{\mathbf{a}}$ in which no variable appears with index larger than $\ell + m$, the inner product

$$\langle \mathbf{a}, \mathbf{w} \rangle = a_1 v_1 + a_2 v_2 + \cdots + a_\ell v_\ell = \langle (a_1, a_2, \ldots, a_\ell), \mathbf{v} \rangle < \infty. \tag{16}$$

For monomials in which there is at least one variable with index larger than $\ell + m$, we have $\langle \mathbf{a}, \mathbf{w} \rangle = \infty$.

In the case where f_i vanishes entirely when the last $n - \ell - m$ variables are zero, every monomial has at least one variable with index larger than $\ell + m$. In that case $\mathrm{in}_{\mathbf{w}}(f_i) = f_i$ and $f_i(\mathbf{z}) = 0$. As $\mathrm{in}_{\mathbf{v}}(0) = 0$, we have that (15) holds for all values t_{D+k}, $k = 1, 2, \ldots, m$. In the other case, there are monomials $\mathbf{x}^{\mathbf{a}}$ in which no variables appear with index larger than $\ell + m$ and for those $\mathbf{x}^{\mathbf{a}}$: $\langle \mathbf{a}, \mathbf{w} \rangle < \infty$. Vanishing monomials have a variable with index larger than $\ell + m$ and $\langle \mathbf{a}, \mathbf{w} \rangle = \infty$. By (16), for monomials that have no variables with index larger than $\ell + m$, $\langle \mathbf{a}, \mathbf{w} \rangle$ equals the inner product with $\mathbf{v}$. Thus (15) holds. $\square$

Proposition 6.3 allows to make the connection with stable mixed volumes. In particular, the inner normals of the stable mixed cells contain the origin that is lifted sufficiently high, leading to some components in the inner normal of much higher magnitude than the others. As in the case of those inner normals, we can extend the tropisms $\mathbf{v}$ of the specialized system to tropisms $\mathbf{w}$ of the original system, where the values that correspond to the variables that are set to zero are sufficiently high.

We end this paper with the observation that although most initial form systems are not binomial, all Puiseux series have a leading term which satisfies a binomial system. The combinatorial algorithms for the defining equations of monomial maps help solving the initial form systems.

7 Computational Experiments

Since version 2.3.68 of PHCpack [37], the black box solver (called as `phc -b`) computes toric solutions of binomial systems. This code is also available via the Python interface `phcpy` [36].

The polynomial equations of adjacent minors are defined in [20, page 631]: $x_{i,j}x_{i+1,j+1} - x_{i+1,j}x_{i,j+1} = 0$, $i = 1, 2, \ldots, m-1$, $j = 1, 2, \ldots, n-1$. For $m = 2$, the solution set is pure dimensional, of dimension $2n - (n-1) = n+1$, the number of irreducible components of X equals the nth Fibonacci number [35, Theorem 5.9], and the degree of the entire solution set is 2^n. Knowing that the solution set is pure dimensional, our enumeration can be restricted so only sets of the right dimension are returned: for every variable we set to zero, one equation has to vanish as well. With this assumption, our enumeration produces exactly the right number of components.

Table 1 shows the comparison between the method proposed in this paper and a witness set construction. This construction requires the computation of as many generic points as the degree of the solution set, which is 2^{n-1} in this case. For $n-1$ quadrics, a total degree homotopy is optimal in the sense that no paths diverge. Experimental results show that the witness set construction has a limited range. In addition, our new method returns the irreducible decomposition.

n	2^{n-1}	#maps	search	witness
3	4	2	0.00	0.03
4	8	3	0.00	0.16
5	16	5	0.00	0.68
6	32	8	0.00	2.07
7	64	13	0.01	7.68
8	128	21	0.01	28.10
9	256	34	0.02	71.80
10	512	55	0.05	206.01
11	1024	89	0.10	525.46
12	2048	144	0.24	—
13	4096	233	0.57	—
14	8192	377	1.39	—
15	16384	610	3.33	—
16	32768	987	8.57	—
17	65536	1597	21.36	—
18	131072	2584	55.95	—
19	262144	4181	140.84	—
20	524288	6765	372.62	—
21	1048576	10946	994.11	—

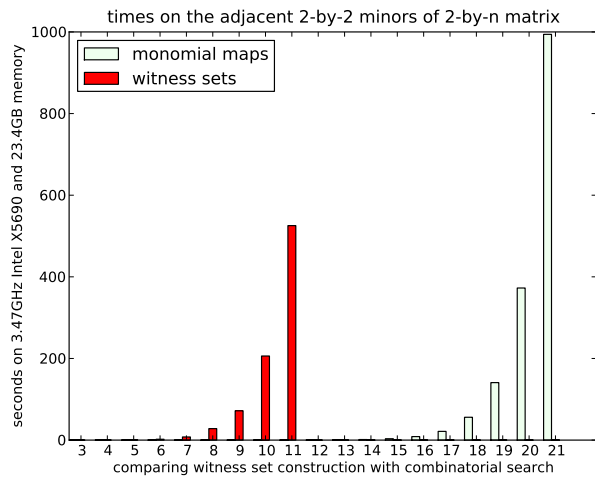times on the adjacent 2-by-2 minors of 2-by-n matrix

Table 1: The construction of a witness set for all adjacent minors of a general 2-by-n matrix requires the tracking of 2^{n-1} paths which is much more expensive than the combinatorial search. For n ranging from 3 to 21 we list times in seconds for the combinatorial search in column 3 and for the witness construction in the last column, capping the time at 1000 seconds.

The system of adjacent minors is also one of the benchmarks in [4], but neither Bertini [5] nor Singular [11] are able to get within the same range of PHCpack. This is not a surprising

conclusion since polyhedral methods scale very well for binomial systems.

In [35, §5.3], the adjacent minors introduce readers to the joys of primary decomposition and the 4-by-4 case is explicitly described in [35, Lemma 5.10] and [35, Theorem 5.11]. Running `phc -b` we see 15 solution maps appear (in agreement with the 15 primes of [35, Lemma 5.10]). Of the 15, 12 maps have dimension 9 and their degrees add up to 32. There are two linear solution sets of dimension 8 and one 7-dimensional solution set of degree 20.

All adjacent 2-by-2 minors of a general 5-by-5 matrix have 100 irreducible components. There are two linear maps of dimension 15, twelve 14-dimensional linear maps, 22 maps of dimension 13 with degrees adding up to 110, the sum of the degrees of the 63 12-dimensional maps equals 582, and finally, there is one 9-dimensional solution set of degree 70. The sum of the degrees of all 100 components equals 776.

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