Tropical Approach to the Cyclic $n$-Roots Problem

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Cyclic n-roots Problem

\[ C_n(x) = \begin{cases} 
  x_0 + x_1 + \cdots + x_{n-1} = 0 \\
  x_0x_1 + x_1x_2 + \cdots + x_{n-2}x_{n-1} + x_{n-1}x_0 = 0 \\
  i = 3, 4, \ldots, n-1 : \sum_{j=0}^{n-1} \prod_{k=j}^{j+i-1} x_k \mod n = 0 \\
  x_0x_1x_2 \cdots x_{n-1} - 1 = 0. 
\end{cases} \]

- benchmark problem in the field of computer algebra (pop. by J. Davenport)
- extremely hard to solve for \( n \geq 8 \)
- square systems
  - we expect isolated solutions
  - we find positive dimensional solution sets

Lemma (Backelin)

*If \( m^2 \) divides \( n \), then the dimension of the cyclic n-roots polynomial system is at least \( m - 1 \).*

J. Backelin: *Square multiples n give infinitely many cyclic n-roots.*
Reports, Matematiska Institutionen, Stockholms Universitet, 1989.
J. Davenport. *Looking at a set of equations.*
Our Approach

a new polyhedral method

- for square systems and systems with more equations than unknowns
- a symbolic-numeric approach with an origin in polyhedral homotopies
- Bernshtein’s Theorem A & B to solve polynomial systems with Puiseux series
- we aim to generalize polyhedral homotopies to develop positive dimensional solution sets
- our approach is inspired by the constructive proof of the fundamental theorem of tropical algebraic geometry

Theorem (Fundamental Theorem of Tropical Algebraic Geometry)

\[ \omega \in Trop(I) \cap \mathbb{Q}^n \iff \exists p \in V(I) : -\text{val}(p) = \omega \in \mathbb{Q}^n. \]


rephrasing the theorem

rational vector in the tropical variety corresponds to the leading powers of a Puiseux series, converging to a point in the algebraic variety.

- we understand the fundamental theorem via polyhedral homotopies
- we see it as a generalization of Bernshtein’s Theorem B
Definition (Polynomial System)

\[ F(x) = \begin{cases} 
  f_0(x) = 0 \\
  f_1(x) = 0 \\
  \vdots \\
  f_{n-1}(x) = 0 
\end{cases} \]

Definition (Laurent Polynomial)

\[ f(x) = \sum_{a \in A} c_a x^a, \quad c_a \in \mathbb{C} \setminus \{0\}, \quad x^a = x_0^{\pm a_0} x_1^{\pm a_1} \cdots x_{n-1}^{\pm a_{n-1}} \]

Definition (Support Set)

The set of exponents \( A_i \) is called the support set of \( f_i \).

Definition (Newton Polytope)

Let \( A_i \) be the support set of the polynomial \( f_i \in F(x) = 0 \). Then, the Newton polytope of \( f_i \) is the convex hull of \( A_i \), denoted \( P_i \).

- equivalent representation of \( P_i \) (or any polytope) in \( \mathbb{R}^n \)
  - convex hull of finite set of points, i.e. V-representation
  - intersection of finitely many closed half-spaces, i.e. H-representation
General Definitions

**Definition (Initial Form)**

Let \( f(x) = \sum_{a \in A} c_a x^a \) be a Laurent polynomial, \( \mathbf{v} \in \mathbb{Z}^n \) a non-zero vector and let \( \langle \cdot, \cdot \rangle \) denote the usual inner product. Then, the **initial form** with respect to \( \mathbf{v} \) is given by

\[
in_{\mathbf{v}}(f(x)) = \sum_{a \in A, \; m = \langle a, \mathbf{v} \rangle} c_a x^a \quad m = \min \{ \langle a, \mathbf{v} \rangle \mid a \in A \}.
\]

**Definition (Initial Form System)**

For a system of polynomials \( \mathbf{F}(x) = 0 \), the **initial form system** is defined by

\[
in_{\mathbf{v}}(\mathbf{F}(x)) = (in_{\mathbf{v}}(f_0), \; in_{\mathbf{v}}(f_1), \ldots, \; in_{\mathbf{v}}(f_{n-1})) = 0.
\]

**Definition (Pretropism)**

A **pretropism** \( \mathbf{v} \in \mathbb{Z}^n \) is a vector, which leads to an initial form system.

**Definition (Tropism)**

A **tropism** is a pretropism, which is the leading exponent vector of a Puiseux series expansion for a curve, expanded about \( t \approx 0 \).
**Definition (Cone of Tropisms)**

A **cone of tropisms** is a polyhedral cone, spanned by tropisms.

- \( v_0, v_1, \ldots, v_{d-1} \) span a \( d \)-dimensional cone of tropisms
- dimension of the cone is the dimension of the solution set
- we obtain the tropisms by using the **Cayley trick**.

Let \( v_0 = (v_{0,1}, v_{0,2}, \ldots, v_{0,n-1}) \), \( v_1 = (v_{1,0}, v_{1,1}, \ldots, v_{1,n-1}) \), \( v_{d-1} = (v_{d-1,0}, v_{d-1,1}, \ldots, v_{d-1,n-1}) \) be \( d \) tropisms. Let \( r_0, r_1, \ldots, r_{n-1} \) be the solutions of the initial form system \( in_{v_0}(in_{v_1}(\cdots in_{v_{d-1}}(F)\cdots))(x) = 0 \).

\( d \) tropisms generate a Puiseux series expansion of a \( d \)-dimensional surface

\[
\begin{align*}
x_0 &= t_0^{v_{0,0}} t_1^{v_{1,0}} \cdots t_{d-1}^{v_{d-1,0}} (r_0 + c(0,0) t_0^{w_{0,0}} + c(1,0) t_1^{w_{1,0}} + \cdots) \\
x_1 &= t_0^{v_{0,1}} t_1^{v_{1,1}} \cdots t_{d-1}^{v_{d-1,1}} (r_1 + c(0,1) t_0^{w_{0,1}} + c(1,1) t_1^{w_{1,1}} + \cdots) \\
x_2 &= t_0^{v_{0,2}} t_1^{v_{1,2}} \cdots t_{d-1}^{v_{d-1,2}} (r_2 + c(0,2) t_0^{w_{0,2}} + c(1,2) t_1^{w_{1,2}} + \cdots) \\
&\vdots \\
x_{n-1} &= t_0^{v_{0,n-1}} t_1^{v_{1,n-1}} \cdots t_{d-1}^{v_{d-1,n-1}} (r_{n-1} + c(0,n-1) t_0^{w_{0,n-1}} + c(1,n-1) t_1^{w_{1,n-1}} + \cdots)
\end{align*}
\]
Cyclic 4-roots problem

Cyclic 4-Root Polynomial System

\[ C_4(x) = \begin{cases} 
  x_0 + x_1 + x_2 + x_3 = 0 \\
  x_0x_1 + x_0x_3 + x_1x_2 + x_2x_3 = 0 \\
  x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3 = 0 \\
  x_0x_1x_2x_3 - 1 = 0 
\end{cases} \]

The only *pretropism* is \((1, -1, 1, -1)\)

cyclic 4-roots initial form system in direction \((1, -1, 1, -1)\)

\[ \text{in}_{(1, -1, 1, -1)}(C_4)(x) = \begin{cases} 
  x_1 + x_3 = 0 \\
  x_0x_1 + x_0x_3 + x_1x_2 + x_2x_3 = 0 \\
  x_0x_1x_3 + x_1x_2x_3 = 0 \\
  x_0x_1x_2x_3 - 1 = 0 
\end{cases} \]

Using \(M\) to transform \(\text{in}_{(1, -1, 1, -1)}(C_4)\):

\[
M = \begin{bmatrix}
1 & -1 & 1 & -1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[ x_0 = z_0; \quad x_1 = \frac{z_1}{z_0}; \quad x_2 = z_0z_2; \quad x_3 = \frac{z_3}{z_0} \]

\[ \text{in}_{(1, -1, 1, -1)}(F)(z) = \begin{cases} 
  z_1/z_0 + z_3/z_0 = 0 \\
  z_1z_2 + z_2z_3 + z_1 + z_3 = 0 \\
  z_1z_2z_3/z_0 + z_1z_3/z_0 = 0 \\
  z_1z_2z_3 - 1 = 0 
\end{cases} \]
Cyclic 4-roots problem

cyclic 4-root polynomial system transformed

\[ \text{in}_{(1,-1,1,-1)}(C_4)(z) = \begin{cases} 
  z_1 + z_3 = 0 \\
  z_1z_2 + z_2z_3 + z_1 + z_3 = 0 \\
  z_1z_2z_3 + z_1z_3 = 0 \\
  z_1z_2z_3 - 1 = 0 
\end{cases} \]

Solutions of the transformed initial form system are
\((z_1 = 1, z_2 = -1, z_3 = -1)\) and \((z_1 = -1, z_2 = -1, z_3 = 1)\).

Letting \(z_0 = t\) and returning solutions to original coordinates with

\[ x_0 = z_0; \quad x_1 = \frac{z_1}{z_0}; \quad x_2 = z_0z_2; \quad x_3 = \frac{z_3}{z_0} \]

For cyclic 4-roots, the initial terms of the series are exact solutions

\[ \begin{cases} 
  x_0 = t^1 \\
  x_1 = t^{-1} \\
  x_2 = -t^1 \\
  x_1 = -t^{-1} 
\end{cases} \quad \text{and} \quad \begin{cases} 
  x_0 = t^1 \\
  x_1 = -t^{-1} \\
  x_2 = -t^1 \\
  x_1 = t^{-1} 
\end{cases} \]
Cyclic 4, 8, 12-roots problem

cyclic 4-roots:
tropism: (1, -1, 1, -1)
\[ x_0 = t, \ x_1 = t^{-1}, \ x_2 = -t, \ x_3 = -t^{-1} \]

cyclic 8-roots:
tropism: (1, -1, 1, -1, 1, -1, 1, -1)
\[ x_0 = t, \ x_1 = t^{-1}, \ x_2 = it, \ x_3 = it^{-1}, \ x_4 = -t, \ x_5 = -t^{-1}, \ x_6 = -it, \ x_7 = -it^{-1} \]

cyclic 12-roots:
tropism: (1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1)
\[ x_0 = t, \ x_1 = t^{-1}, \ x_2 = \left(\frac{1 + \sqrt{3}i}{2}\right)t, \ x_3 = \left(\frac{1 + \sqrt{3}i}{2}\right)t^{-1}, \ x_4 = \left(-\frac{1 + \sqrt{3}i}{2}\right)t, \ x_5 = \left(-\frac{1 + \sqrt{3}i}{2}\right)t^{-1}, \ x_6 = -t, \ x_7 = -t^{-1}, \ x_8 = \left(-\frac{1 - \sqrt{3}i}{2}\right)t, \ x_9 = \left(-\frac{1 - \sqrt{3}i}{2}\right)t^{-1}, \ x_{10} = \left(\frac{1 - \sqrt{3}i}{2}\right)t, \ x_{11} = \left(\frac{1 - \sqrt{3}i}{2}\right)t^{-1} \]

Observing structure among
- tropism
- coefficients
  - numerical solver PHCpack was used
  - we recognize the coefficients as $\frac{n}{2}$-roots of unity
Proposition

For $n = 4m$, there is a one-dimensional set of cyclic $n$-roots, represented exactly as

\begin{align*}
x_{2k} &= u_k t \\
x_{2k+1} &= u_k t^{-1}
\end{align*}

for $k = 0, \ldots, \frac{n}{2} - 1$ and $u_k = e^{\frac{i2\pi k}{n}} = e^{\frac{i4\pi k}{n}}$.

taking random linear combination of the solutions

$$
\alpha_0 t + \alpha_1 t^{-1} + \alpha_2 t + \alpha_3 t^{-1} + \cdots + \alpha_{n-2} t + \alpha_{n-1} t^{-1} = 0, \quad \alpha_j \in \mathbb{C}
$$

and simplifying

$$
\beta_0 t^2 + \beta_1 = 0, \quad \beta_j \in \mathbb{C}
$$

we see that all space curves are **quadrics**.
The cone of pretropisms for the cyclic 9-roots polynomial system was generated by pretropism sequence
\[ v_0 = (1, 1, -2, 1, 1, -2, 1, 1, -2) \]
\[ v_1 = (0, 1, -1, 0, 1, -1, 0, 1, -1). \]

\[
\ln_{v_1}(\ln_{v_0}(C_9))(x) = \begin{cases} 
  x_2 + x_5 + x_8 = 0 \\
  x_0x_8 + x_2x_3 + x_5x_6 = 0 \\
  x_0x_1x_2 + x_0x_1x_8 + x_0x_7x_8 + x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 \\
  + x_4x_5x_6 + x_5x_6x_7 + x_6x_7x_8 = 0 \\
  x_0x_1x_2x_8 + x_2x_3x_4x_5 + x_5x_6x_7x_8 = 0 \\
  x_0x_1x_2x_3x_8 + x_0x_5x_6x_7x_8 + x_2x_3x_4x_5x_6 = 0 \\
  x_0x_1x_2x_3x_4x_5 + x_0x_1x_2x_3x_4x_8 + x_0x_1x_2x_3x_7x_8 \\
  + x_0x_1x_2x_6x_7x_8 + x_0x_1x_5x_6x_7x_8 + x_0x_4x_5x_6x_7x_8 + x_1x_2x_3x_4x_5x_6 \\
  + x_2x_3x_4x_5x_6x_7 + x_3x_4x_5x_6x_7x_8 = 0 \\
  x_0x_1x_2x_3x_4x_5x_8 + x_0x_1x_2x_5x_6x_7x_8 + x_2x_3x_4x_5x_6x_7x_8 = 0 \\
  x_0x_1x_2x_3x_4x_5x_6x_8 + x_0x_1x_2x_3x_5x_6x_7x_8 + x_0x_2x_3x_4x_5x_6x_7x_8 = 0 \\
  x_0x_1x_2x_3x_4x_5x_6x_7x_8 - 1 = 0 
\end{cases}
\]

Cyclic 9-Roots Polynomial System Cont.

\[ \nu_0 = (1, 1, -2, 1, 1, -2, 1, 1, -2) \]
\[ \nu_1 = (0, 1, -1, 0, 1, -1, 0, 1, -1) \]

The unimodular coordinate transformation \( x = z^M \) acts on the exponents. The new coordinates are given by

\[
M = \begin{bmatrix}
1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & -2 \\
0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[ x_0 = z_0 \]
\[ x_1 = z_0 z_1 \]
\[ x_2 = z_0^{-2} z_1^{-1} z_2 \]
\[ x_3 = z_0 z_3 \]
\[ x_4 = z_0 z_1 z_4 \]
\[ x_5 = z_0^{-2} z_1^{-1} z_5 \]
\[ x_6 = z_0 z_6 \]
\[ x_7 = z_0 z_1 z_7 \]
\[ x_8 = z_0^{-2} z_1^{-1} z_8 \]

We use the coordinate change to transform the initial form system and the original cyclic 9-roots system.
The transformed initial form system $\text{inv}_1(\text{inv}_0(C_9))(z)$ is given by

$$
\begin{align*}
  z_2 + z_5 + z_8 &= 0 \\
  z_2z_3 + z_5z_6 + z_8 &= 0 \\
  z_2z_3z_4 + z_3z_4z_5 + z_4z_5z_6 + z_5z_6z_7 + z_6z_7z_8 + z_2z_3 + z_7z_8 + z_2 + z_8 &= 0 \\
  z_2z_3z_4z_5 + z_5z_6z_7z_8 + z_2z_8 &= 0 \\
  z_2z_3z_4z_5z_6 + z_5z_6z_7z_8 + z_2z_3z_8 &= 0 \\
  z_2z_3z_4z_5z_6z_7 + z_3z_4z_5z_6z_7z_8 + z_2z_3z_4z_5z_6 + z_4z_5z_6z_7z_8 + z_2z_3z_4z_5 + z_2z_3z_4z_8 + z_2z_3z_7z_8 + z_2z_6z_7z_8 + z_5z_6z_7z_8 &= 0 \\
  z_3z_4z_6z_7 + z_3z_4 + z_6z_7 &= 0 \\
  z_4z_7 + z_4 + z_7 &= 0 \\
  z_2z_3z_4z_5z_6z_7z_8 - 1 &= 0
\end{align*}
$$

Its solution is

$$
\begin{align*}
  z_2 &= -\frac{1}{2} - \frac{\sqrt{3}i}{2}, \\
  z_3 &= -\frac{1}{2} + \frac{\sqrt{3}i}{2}, \\
  z_4 &= -\frac{1}{2} + \frac{\sqrt{3}i}{2}, \\
  z_5 &= 1, \\
  z_6 &= -\frac{1}{2} - \frac{\sqrt{3}i}{2}, \\
  z_7 &= -\frac{1}{2} - \frac{\sqrt{3}i}{2}, \\
  z_8 &= -\frac{1}{2} + \frac{\sqrt{3}i}{2},
\end{align*}
$$

where $i = \sqrt{-1}$.

While we used a numerical solver PHCpack, we recognized the solution as the $3^{rd}$ roots of unity.
The following assignment satisfies cyclic 9-roots polynomial system entirely.

\[ z_0 = t_0 \]
\[ z_1 = t_1 \]
\[ z_2 = -\frac{1}{2} - \frac{\sqrt{3}i}{2} \]
\[ z_3 = -\frac{1}{2} + \frac{\sqrt{3}i}{2} \]
\[ z_4 = -\frac{1}{2} + \frac{\sqrt{3}i}{2} \]
\[ z_5 = 1 \]
\[ z_6 = -\frac{1}{2} - \frac{\sqrt{3}i}{2} \]
\[ z_7 = -\frac{1}{2} - \frac{\sqrt{3}i}{2} \]
\[ z_8 = -\frac{1}{2} + \frac{\sqrt{3}i}{2} \]
Letting $u = e^{\frac{2\pi i}{3}}$ and $y_0 = t_0$, $y_1 = t_0 t_1$, $y_2 = t_0^{-2} t_1^{-1} u^2$
we can rewrite the exact solution as

\[
\begin{align*}
  x_0 &= t_0 & x_3 &= t_0 u & x_6 &= t_0 u^2 \\
  x_1 &= t_0 t_1 & x_4 &= t_0 t_2 u & x_7 &= t_0 t_2 u^2 \\
  x_2 &= t_0^{-2} t_1^{-1} u^2 & x_5 &= t_0^{-2} t_1^{-1} & x_8 &= t_0^{-2} t_1^{-1} u \\
\end{align*}
\]

and put it in the same format as in the proof of Backelin’s Lemma, given in
J. C. Faugère, *Finding all the solutions of Cyclic 9 using Gröbner basis techniques.*
In Computer Mathematics: Proceedings of the Fifth Asian Symposium (ASCM),

degree of the solution component

\[
\begin{align*}
  \alpha_1 t_0 + \alpha_2 t_0 t_1 + \alpha_3 t_0^{-2} t_1^{-1} &= 0 \\
  \alpha_4 t_0 + \alpha_5 t_0 t_1 + \alpha_6 t_0^{-2} t_1^{-1} &= 0 \\
\end{align*}
\]

$\alpha_i \in \mathbb{C}$

Simplifying, the system becomes

\[
\begin{align*}
  t_0^{-2} t_1^{-1} - \beta_1 &= 0 & \text{forwards} : & [1, u, u^2] \to [u, u^2, 1] \to [u^2, 1, u] \\
  t_1 - \beta_2 &= 0 & \text{backwards} : & [u^2, u, 1] \to [u, 1, u^2] \to [1, u^2, u] \\
\end{align*}
\]

As the simplified system has 3 solutions, the cyclic 9 solution component
is a **cubic** surface. With the cyclic permutation, we obtain an orbit of 6
cubic surfaces, which satisfy the cyclic 9-roots system.
Extending the pattern we observed among tropisms of the cyclic 9-roots, 
\( v_0 = (1, 1, -2, 1, 1, -2, 1, 1, -2) \)
\( v_1 = (0, 1, -1, 0, 1, -1, 0, 1, -1) \)
we can get the correct cone of tropisms for the cyclic 16-roots.
\( v_0 = (1, 1, 1, -3, 1, 1, 1, -3, 1, 1, 1, -3) \)
\( v_1 = (0, 1, 1, -2, 0, 1, 1, -2, 0, 1, 1, -2) \)
\( v_2 = (0, 0, 1, -1, 0, 0, 1, -1, 0, 0, 1, -1) \)
Extending the solutions at infinity pattern,
cyclic 9-roots: \( u = e^{\frac{2\pi i}{3}} \) \to cyclic 16-roots: \( u = e^{\frac{2\pi i}{4}} \)
The 3-dimensional solution component of the cyclic 16-roots is given by:

\[
\begin{align*}
x_0 &= t_0 \\
x_1 &= t_0 t_1 \\
x_2 &= t_0 t_1 t_2 \\
x_3 &= t_0^{-3} t_1^{-2} t_2^{-1} \\
x_4 &= u t_0 \\
x_5 &= u t_0 t_1 \\
x_6 &= u t_0 t_1 t_2 \\
x_7 &= u t_0^{-3} t_1^{-2} t_2^{-1} \\
x_8 &= u^2 t_0 \\
x_9 &= u^2 t_0 t_1 \\
x_{10} &= u^2 t_0 t_1 t_2 \\
x_{11} &= u^2 t_0^{-3} t_1^{-2} t_2^{-1} \\
x_{12} &= u^3 t_0 \\
x_{13} &= u^3 t_0 t_1 \\
x_{14} &= u^3 t_0 t_1 t_2 \\
x_{15} &= u^3 t_0^{-3} t_1^{-2} t_2^{-1}
\end{align*}
\]

This 3-dimensional cyclic 16-root solution component is a **quartic** surface.
Using cyclic permutation, we obtain \( 2 \times 4 = 8 \) components of degree 4.
We now generalize the previous results for the cyclic n-roots systems.

**Proposition** For \( n = m^2 \), there is a \((m - 1)\)-dimensional set of cyclic n-roots, represented exactly as

\[
\begin{align*}
x_{km+0} &= u_k t_0 \\
x_{km+1} &= u_k t_0 t_1 \\
x_{km+2} &= u_k t_0 t_1 t_2 \\
&\vdots \\
x_{km+m-2} &= u_k t_0 t_1 t_2 \cdots t_{m-2} \\
x_{km+m-1} &= u_k t_0^{-m+1} t_1^{-m+2} \cdots t_{m-3} t_{m-2}
\end{align*}
\]

for \( k = 0, 1, 2, \ldots, m - 1 \) and \( u_k = e^{i2k\pi/m} \).

**Proposition** The \((m - 1)\)-dimensional solutions set has degree equal to \( m \).

Applying cyclic permutation, we can find \( 2m \) components of degree \( m \).