Polyhedral Methods for Space Curves and Two Dimensional Surfaces Exploiting Symmetry

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Joint Mathematics Meetings
- AMS Session on Algebraic Geometry II -
Boston, Massachusetts

Introduction

goal: use polyhedral methods and Puiseux series to solve systems of polynomials

- space curves
- surfaces

main objects

- tropism
- Puiseux series

focus: **exploitation of symmetry**

illustration on cyclic n-roots benchmark problems

- cyclic n-roots polynomial systems, $n = 4,\ 8,\ 9,\ 12,\ 24$

assumptions on the space curves/surfaces we can find:

- they are reduced, free of multiplicities
- in general position with respect to
  - $x_1 = 0$ (space curves)
  - $x_1 = 0$, $x_2 = 0$ (2D surfaces)

Fundamental Theorem

Theorem (Fundamental Theorem Of Tropical Algebraic Geometry)

$\omega \in Trop(I) \cap \mathbb{Q}^n \iff \exists p \in V(I) : -val(p) = \omega \in \mathbb{Q}^n.$

Rephrasing the Theorem:

Every rational vector in the tropical variety corresponds to the leading powers of a Puiseux series, converging to a point in the algebraic variety.

For a constructive proof of the Fundamental Theorem, we refer to
Anders Nedergaard Jensen, Hannah Markwig, Thomas Markwig:
An Algorithm for Lifting Points in a Tropical Variety.
Collect. Math. vol. 59, no. 2, pages 129–165, 2008.

We see *Fundamental Theorem of Tropical Algebraic Geometry* as a generalization of *Bernshtein's Theorem B*.
We use *Bernshtein's Theorem A &B* as a way to solve polynomial systems with polyhedral methods.

Cyclic n-roots Polynomial Systems

The cyclic n-roots polynomial systems are benchmark problems for polynomial system solvers.

$$F(\mathbf{x}) = \begin{cases} f_1 = x_0 + x_1 + \cdots + x_{n-1} = 0 \\ f_2 = x_0 x_1 + x_1 x_2 + \cdots + x_{n-2} x_{n-1} + x_{n-1} x_0 = 0 \\ i = 3, 4, \ldots, n-1 : \displaystyle\sum_{j=0}^{n-1} \prod_{k=j}^{i} x_{k \mod n} = 0 \\ f_n = x_0 x_1 x_2 \ldots x_{n-1} - 1 = 0 \end{cases}$$

cyclic n-roots polynomial systems:

- square systems: we expect isolated solutions
- for cyclic 4, 8, 12, 24-roots, we have space curves
- for cyclic 9-roots we have a two dimensional surface

J. Backelin: *Square multiples n give infinitely many cyclic n-roots*. Reports, Matematiska Institutionen, Stockholms Universitet, 1989.

Cyclic n-roots Polynomial Systems

Lemma (Backelin)

If m^2 divides n, then the dimension of the cyclic n-roots polynomial system is at least $m - 1$.

J. Backelin: *Square multiples n give infinitely many cyclic n-roots*. Reports, Matematiska Institutionen, Stockholms Universitet, 1989.

Basic Definitions

Polynomial System

$$F(\mathbf{x}) = \begin{cases} f_1 = 0 \\ f_2 = 0 \\ \vdots \\ f_k = 0 \end{cases} \qquad \mathbf{x} = (x_1, x_2, \ldots, x_n), \quad f_i \in \mathbb{C}[\mathbf{x}]$$

We define the *support sets* of $\mathrm{F}(\mathbf{x}) = 0$ to be:

$$\mathbf{(}A_1, A_2, \ldots, A_k) = (Supp(f_1), Supp(f_2), \ldots, Supp(f_k))$$

with $A_i = ((a_{11}, a_{12}, \ldots, a_{1n}), (a_{21}, a_{22}, \ldots, a_{2n}), \ldots, (a_{N1}, a_{N2}, \ldots, a_{Nn}))$, where N is the number of monomials in f_i, $a_j \in \mathbb{Z}$ and $(a_{j1}, a_{j2}, \ldots, a_{jn})$ are exponents of a monomial in f_i.

Support set A_i of f_i spans the *Newton polytope* as $P_i = ConvexHull(A_i)$

The Cayley Embedding & Polytope

Cayley embedding C_E of the set **A**

$$C_E(A_1, A_2, \ldots A_k) = (A_1 \times \{0\}_{k-1}) \cup \bigcup_{i=1}^{k-1} (A_{i+1} \times e_i)$$

where e_i denotes the i^{th} vector of the standard basis.

Cayley polytope

$$C_\Delta = conv(C_E) \ \subset \mathbb{R}^{n+k-1}$$

NOTE

We use the Cayley polytope as a way to combine all individual polytopes into one polytope.

We use **cddlib** of *K. Fukuda* to find facet normals of the Cayley polytope. We will refer to the facet normals as **pretropisms**.

Tropisms and Space Curves

Definition (Pretropism)

A **pretropism** is a normal vector to at least an edge of each polytope.

a pretropism might generate a Puiseux series expansion of a space curve

Let $V = (v_1, v_2, \ldots, v_n)$ be a *pretropism*, $w_i > 0$, b_i, $c_i \in \mathbb{C}$:

$$G(\mathbf{x}, t) = \begin{cases} x_1 = t^{v_1}(b_1 + c_1 t^{w_1} + \ldots) \\ x_2 = t^{v_2}(b_2 + c_2 t^{w_2} + \ldots) \\ x_3 = t^{v_3}(b_3 + c_3 t^{w_3} + \ldots) \\ \vdots \\ x_n = t^{v_n}(b_n + c_n t^{w_n} + \ldots) \end{cases}$$

Definition (Tropism)

A **tropism** is a pretropism which is the leading exponent vector of a Puiseux series expansion for a curve, expanded about $t = 0$.

Initial Ideal

Definition (Initial Form)

Let f be a polynomial with support A and let V be a pretropism. Then the **initial form** $in_V(f)$ is the sum of all monomials in f, where the inner product $\langle \mathbf{a}, V \rangle$ reaches its minimum at least twice over $\mathbf{a} \in A$.

Initial Form System

For a system $F(\mathbf{x}) = \mathbf{0}$, $F = (f_1, f_2, \ldots, f_k)$, and pretropism V, the **initial form system** is defined by $in_V(F) = (in_V(f_1), in_V(f_2), \ldots, in_V(f_k))$.

Solving initial form system leads to solutions at infinity that are isolated, or to the leading coefficients of the Puiseux expansion of a curve.

Puiseux series expansion of a curve: solutions at infinity, denoted by b_i

Let $V = (v_1, v_2, \ldots, v_n)$ be a *pretropism* and let t denote a free variable:

$$x_i = t^{v_i}(b_i + c_i t^{w_i} + \cdots), \quad i = 1, 2, \ldots, n.$$

Cyclic 4-Roots System

The only *pretropism* is $(1, -1, 1, -1)$

Cyclic 4-Roots Initial Form In Direction $(1, -1, 1, -1)$

$$in_{(1,-1,1,-1)}(F)(\mathbf{x}) = \begin{cases} x_1 + x_3 = 0 \\ x_0 x_1 + x_0 x_3 + x_1 x_2 + x_2 x_3 = 0 \\ x_0 x_1 x_3 + x_1 x_2 x_3 = 0 \\ x_0 x_1 x_2 x_3 - 1 = 0 \end{cases}$$

Using U to transform $in_{(1,-1,1,-1)}(F)$:

$$U = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x_0 = z_0; \; x_1 = \frac{z_1}{z_0}; \; x_2 = z_0 z_2; \; x_3 = \frac{z_3}{z_0}$$

$$in_{(1,-1,1,-1)}(F)(\mathbf{z}) = \begin{cases} z_1/z_0 + z_3/z_0 = 0 \\ z_1 z_2 + z_2 z_3 + z_1 + z_3 = 0 \\ z_1 z_2 z_3/z_0 + z_1 z_3/z_0 = 0 \\ z_1 z_2 z_3 - 1 = 0 \end{cases}$$

Cyclic 4-Roots System

Cyclic 4-Root Polynomial System Transformed

$$in_{(1,-1,1,-1)}(F)(\mathbf{z}) = \begin{cases} z_1 + z_3 = 0 \\ z_1 z_2 + z_2 z_3 + z_1 + z_3 = 0 \\ z_1 z_2 z_3 + z_1 z_3 = 0 \\ z_1 z_2 z_3 - 1 = 0 \end{cases}$$

Solutions of the transformed initial form system are $(z_1 = 1, z_2 = -1, z_3 = -1)$ and $(z_1 = -1, z_2 = -1, z_3 = 1)$. Let $z_0 = t$:

For cyclic 4-roots, the initial terms of the series are exact solutions

$$\begin{cases} x_0 = t^1 \\ x_1 = t^{-1} \\ x_2 = -t^1 \\ x_1 = -t^{-1} \end{cases} \quad \textit{and} \quad \begin{cases} x_0 = t^1 \\ x_1 = -t^{-1} \\ x_2 = -t^1 \\ x_1 = t^{-1} \end{cases}$$

Cyclic 8-Roots System

Cyclic 8-roots system:

- 831 facet normals (computed with `cddlib`)
- 29 pretropism generators
- 5 lead to initial forms with solutions
  - $(1, -1, 0, 1, 0, 0, -1, 0)$
  - $(1, -1, 1, -1, 1, -1, 1, -1)$
  - $(1, 0, -1, 0, 0, 1, 0, -1)$
  - $(1, 0, -1, 1, 0, -1, 0, 0)$
  - $(1, 0, 0, -1, 0, 1, -1, 0)$

For the initial form solutions we used the blackbox solver of PHCpack. Symbolic manipulations for the computation of the second term of the Puiseux series were done with Sage.

Cyclic 8-Roots System

For the pretropism $V = (1, -1, 0, 1, 0, 0, -1, 0)$, the initial form system is

$$in_V(F)(\mathbf{x}) = \begin{cases} x_1 + x_6 = 0 \\ x_1x_2 + x_5x_6 + x_6x_7 = 0 \\ x_4x_5x_6 + x_5x_6x_7 = 0 \\ x_0x_1x_6x_7 + x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_6x_7 + x_0x_1x_5x_6x_7 = 0 \\ x_0x_1x_2x_5x_6x_7 + x_0x_1x_4x_5x_6x_7 + x_1x_2x_3x_4x_5x_6 = 0 \\ x_0x_1x_2x_4x_5x_6x_7 + x_1x_2x_3x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_3x_4x_5x_6x_7 - 1 = 0 \end{cases}$$

V defines the unimodular coordinate transformation: $x_0 = z_0$, $x_1 = z_1/z_0, x_2 = z_2, x_3 = z_0z_3, x_4 = z_4, x_5 = z_5, x_6 = z_6/z_0, x_7 = z_7$. Using the new coordinates, we transform the initial form system $in_V(F)(\mathbf{x})$.

Cyclic 8-Roots System

$$in_V(F)(\mathbf{z}) = \begin{cases} z_1 + z_6 = 0 \\ z_1 z_2 + z_5 z_6 + z_6 z_7 = 0 \\ z_4 z_5 z_6 + z_5 z_6 z_7 = 0 \\ z_4 z_5 z_6 z_7 + z_1 z_6 z_7 = 0 \\ z_1 z_2 z_6 z_7 + z_1 z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 z_6 + z_1 z_2 z_5 z_6 z_7 + z_1 z_4 z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 z_6 z_7 + z_1 z_2 z_4 z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 z_6 z_7 - 1 = 0 \end{cases}$$

Solving $in_V(F)(\mathbf{z})$, we obtain 8 solutions (all in the same orbit). We select

$$z_0 = t, z_1 = -I, z_2 = \frac{-1}{2} - \frac{I}{2}, z_3 = -1, z_4 = 1 + I,$$
$$z_5 = \frac{1}{2} + \frac{I}{2}, z_6 = I, z_7 = -1 - I, I = \sqrt{-1}.$$

Cyclic 8-Roots System

Taking solution at infinity, we build a series of the form:

$$z0 = t$$
$$z1 = -I + c_1 t$$
$$z2 = \frac{-1}{2} - \frac{I}{2} + c_2 t$$
$$z3 = -1 + c_3 t$$
$$z4 = 1 + I + c_4 t$$
$$z5 = \frac{1}{2} + \frac{I}{2} + c_5 t$$
$$z6 = I + c_6 t$$
$$z7 = (-1 - I) + c_7 t$$

Plugging series form into transformed system, collecting all coefficients of t^1, solving yields

$$c_1 = -1 - I$$
$$c_2 = \frac{1}{2}$$
$$c_3 = 0$$
$$c_4 = -1$$
$$c_5 = \frac{-1}{2}$$
$$c_6 = 1 + I$$
$$c_7 = 1$$

The second term in the series, still in the transformed coordinates:

$$z0 = t$$
$$z1 = -I + (-1 - I)t$$
$$z2 = \frac{-1}{2} - \frac{I}{2} + \frac{1}{2}t$$
$$z3 = -1$$
$$z4 = 1 + I - t$$
$$z5 = \frac{1}{2} + \frac{I}{2} - \frac{1}{2}t$$
$$z6 = I + (1 + I)t$$
$$z7 = (-1 - I) + t$$

Cyclic 8-Roots System

In certain instances one term in the Puiseux series satisfies the entire system. The initial form in direction $V = (1, -1, 1, -1, 1, -1, 1, -1)$ is $in_V(F)(\mathbf{x}) =$

$$\begin{cases} x_1 + x_3 + x_5 + x_7 = 0 \\ x_0x_1 + x_0x_7 + x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_6 + x_6x_7 = 0 \\ x_0x_1x_7 + x_1x_2x_3 + x_3x_4x_5 + x_5x_6x_7 = 0 \\ x_0x_1x_2x_3 + x_0x_1x_2x_7 + x_0x_1x_6x_7 + x_0x_5x_6x_7 \\ + x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_6 + x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_3x_7 + x_0x_1x_5x_6x_7 + x_1x_2x_3x_4x_5 + x_3x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_3x_4x_5 + x_0x_1x_2x_3x_4x_7 + x_0x_1x_2x_3x_6x_7 + x_0x_1x_2x_5x_6x_7 \\ + x_0x_1x_4x_5x_6x_7 + x_0x_3x_4x_5x_6x_7 + x_1x_2x_3x_4x_5x_6 + x_2x_3x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_3x_4x_5x_7 + x_0x_1x_2x_3x_5x_6x_7 + x_0x_1x_3x_4x_5x_6x_7 + x_1x_2x_3x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_3x_4x_5x_6x_7 - 1 = 0 \end{cases}$$

Cyclic 8-Roots System

The unimodular matrix

$$U = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and its corresponding coordinate transformation:

$$x_0 = z_0, x_1 = z_1/z_0, x_2 = z_0 z_2, x_3 = z_3/z_0,$$
$$x_4 = z_0 z_4, x_5 = z_5/z_0, x_6 = z_0 z_6, x_7 = z_7/z_0.$$

Cyclic 8-Roots System

Initial form for $V = (1, -1, 1, -1, 1, -1, 1, -1)$ after transformation, $in_V(F)(\mathbf{z})$

$$= \begin{cases} z_1 + z_3 + z_5 + z_7 = 0 \\ z_1 z_2 + z_2 z_3 + z_3 z_4 + z_4 z_5 + z_5 z_6 + z_6 z_7 + z_1 + z_7 = 0 \\ z_1 z_2 z_3 + z_3 z_4 z_5 + z_5 z_6 z_7 + z_1 z_7 = 0 \\ z_1 z_2 z_3 z_4 + z_2 z_3 z_4 z_5 + z_3 z_4 z_5 z_6 + z_4 z_5 z_6 z_7 + z_1 z_2 z_3 \\ + z_1 z_2 z_7 + z_1 z_6 z_7 + z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 + z_3 z_4 z_5 z_6 z_7 + z_1 z_2 z_3 z_7 + z_1 z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 z_6 + z_2 z_3 z_4 z_5 z_6 z_7 + z_1 z_2 z_3 z_4 z_5 + z_1 z_2 z_3 z_4 z_7 \\ + z_1 z_2 z_3 z_6 z_7 + z_1 z_2 z_5 z_6 z_7 + z_1 z_4 z_5 z_6 z_7 + z_3 z_4 z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 z_6 z_7 + z_1 z_2 z_3 z_4 z_5 z_7 + z_1 z_2 z_3 z_5 z_6 z_7 + z_1 z_3 z_4 z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 z_6 z_7 - 1 = 0 \end{cases}$$

We then solve $in_V(F)(\mathbf{z})$.

Cyclic 8-Roots System

Initial form system $in_V(F)(\mathbf{z})$ has 72 solutions. In particular

$$
\begin{aligned}
z_0 &= t & \qquad & & x_0 &= t \\
z_1 &= -1 & & & x_1 &= -1/t \\
z_2 &= I & & & x_2 &= It \\
z_3 &= -I & & I = \sqrt{-1} & x_3 &= -I/t \\
z_4 &= -1 & & & x_4 &= -t \\
z_5 &= 1 & & & x_5 &= 1/t \\
z_6 &= -I & & & x_6 &= -It \\
z_7 &= I & & & x_7 &= I/t
\end{aligned}
$$

satisfies the entire cyclic 8-roots polynomial system.
Applying symmetry, we can find the remaining 7 as well.

Cyclic 8-Roots System

Definition (Branch Degree)

Let $V = (v_1, v_2, \ldots, v_m)$ be a tropism and let R be the set of initial roots of the initial form system $in_V(F)(\mathbf{z})$. Then the degree of the branch is

$$\#R \times |\max_{i=1}^{m} v_i - \min_{i=1}^{m} v_i|$$

Tropisms, their cyclic permutations, and degrees:

$$\begin{array}{rr}
(1,-1,1,-1,1,-1,1,-1) & 8 \times 2 = 16 \\
(1,-1,0,1,0,0,-1,0) \rightarrow (1,0,0,-1,0,1,-1,0) & 8 \times 2 + 8 \times 2 = 32 \\
(1,0,-1,0,0,1,0,-1) \rightarrow (1,0,-1,1,0,-1,0,0) & 8 \times 2 + 8 \times 2 = 32 \\
(1,0,-1,1,0,-1,0,0) \rightarrow (1,0,-1,0,0,1,0,-1) & 8 \times 2 + 8 \times 2 = 32 \\
(1,0,0,-1,0,1,-1,0) \rightarrow (1,-1,0,1,0,0,-1,0) & 8 \times 2 + 8 \times 2 = 32 \\
 & \text{TOTAL} = 144
\end{array}$$

144 is the degree of the solution curve of the cyclic 8-root system.

The only tropisms is $V = (1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1)$.
The generating solutions to the quadratic space curve solutions of the cyclic 12-roots problem are on the next slide.
They are given in the transformed coordinates.
For any solution generator $(r_1, r_2, \dots, r_{11})$:

$$z_0 = t,\ z_1 = r_1,\ z_2 = r_2,\ z_3 = r_3,\ z_4 = r_4,\ z_5 = r_5,$$
$$z_6 = r_6,\ z_7 = r_7,\ z_8 = r_8,\ z_9 = r_9,\ z_{10} = r_{10},\ z_{11} = r_{11}$$

we return it to the original coordinates we obtain

$$x_0 = t,\ x_1 = \frac{r_1}{t},\ x_2 = r_2 t,\ x_3 = \frac{r_3}{t},\ x_4 = r_4 t,\ x_5 = \frac{r_5}{t}$$
$$x_6 = r_6 t,\ x_7 = \frac{r_7}{t},\ x_8 = r_8 t,\ x_9 = \frac{r_9}{t},\ x_{10} = r_{10} t,\ x_{11} = \frac{r_{11}}{t}$$

Applying definition for the branch degree,

$$\#R \times |\max_{i=1}^{m} v_i - \min_{i=1}^{m} v_i|,$$

we see that all space curves are quadric.

R. Sabeti. *Numerical-symbolic exact irreducible decomposition of cyclic-12*. LMS Journal of Computation and Mathematics, 14:155172, 2011.

Generators of the roots of the initial form system $In_V(C_{12})(\mathbf{z}) = 0$.

z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}
$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$
$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	1	1	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	-1	-1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$
$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}$
$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	-1	-1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	1	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	1
$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2}$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	1	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$
1	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	-1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	1	-1	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$
$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$
$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	1	-1	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	-1
1	-1	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	-1	1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$
$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2}$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$
$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	-1	$\frac{1}{2}$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1
$-\frac{1}{2}$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	1	-1	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	-1
$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$
$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$
$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	-1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	-1	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	1	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1
$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$
1	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	-1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$
-1	-1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$
-1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	-1	1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	-1	-1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$
$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$	1	$-\frac{1}{2}$	-1	$\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}I$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2}-\frac{\sqrt{3}}{2}I$

Extending the tropism $V_{12} = (1,-1,1,-1,1,-1,1,-1,1,-1,1,-1)$ of the cyclic 12-roots polynomial system to $V_{24} =$ $(1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,-1)$, we obtain a valid tropisms for the cyclic 24-roots.
Exploiting the symmetry of the solution generators of the cyclic 12, we can solve the cyclic 24-roots in direction of V_{24} and obtain exact representation of one of its components!

$x_0 = t,\ x_1 = t^{-1}(\frac{\sqrt{3}}{2} + \frac{I}{2}),\ x_2 = t(\frac{1}{2} - \frac{\sqrt{3}I}{2}),\ x_3 = t^{-1}(-\frac{\sqrt{3}}{2} + \frac{I}{2}),$
$x_4 = -t,\ x_5 = t^{-1}(\frac{\sqrt{3}}{2} - \frac{I}{2}),\ x_6 = t,\ x_7 = t^{-1}(-\frac{\sqrt{3}}{2} - \frac{I}{2}),$
$x_8 = t(1.86602540378444 - 3.23205080756888 * I),$
$x_9 = t^{-1}(0.232050807568877 - 0.133974596215561 * I),$
$x_{10} = -t,\ x_{11} = t^{-1}(-\frac{\sqrt{3}}{2} + \frac{I}{2}),\ x_{12} = -t,\ x_{13} = t^{-1}(-\frac{\sqrt{3}}{2} - \frac{I}{2}),$
$x_{14} = t(-\frac{1}{2} + \frac{\sqrt{3}I}{2}),\ x_{15} = t^{-1}(\frac{\sqrt{3}}{2} - \frac{I}{2}),\ x_{16} = t,\ x_{17} = t^{-1}(-\frac{\sqrt{3}}{2} + \frac{I}{2}),$
$x_{18} = -t,\ x_{19} = t^{-1}(\frac{\sqrt{3}}{2} + \frac{I}{2}),$
$x_{20} = t(-1.86602540378444 + 3.23205080756888 * I),$
$x_{21} = t^{-1}(-0.232050807568877 + 0.133974596215561 * I),$
$x_{22} = t,\ x_{23} = t^{-1}(\frac{\sqrt{3}}{2} - \frac{I}{2}),$

Let $U = (u_1, u_2, \ldots, u_n)$ and $V = (v_1, v_2, \ldots, v_n)$ be two tropisms:

a pair of tropisms generate a Puiseux series expansion of a two-dimensional surface

$$G(\mathbf{x}, t_1,\ t_2) = \begin{cases} x_1 & = t_1^{u_1} t_2^{v_1} (b_1 + c_1 t_1^{\alpha_1} + d_1 t_2^{\beta_1} + \ldots) \\ x_2 & = t_1^{u_2} t_2^{v_2} (b_2 + c_2 t_1^{\alpha_2} + d_2 t_2^{\beta_2} + \ldots) \\ x_3 & = t_1^{u_3} t_2^{v_3} (b_3 + c_3 t_1^{\alpha_3} + d_3 t_2^{\beta_3} + \ldots) \\ & \vdots \\ x_n & = t_1^{u_n} t_2^{v_n} (b_n + c_n t_1^{\alpha_n} + d_n t_2^{\beta_n} + \ldots) \end{cases}$$

Tropisms and Two-Dimensional Surfaces

Square matrix M, with det(M) = ± 1, is composed of two tropisms U and V and integer values, denoted with $*$.

$$M = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & \cdots & u_{n-1} & u_n \\ v_1 & v_2 & v_3 & v_4 & \cdots & v_{n-1} & v_n \\ * & * & * & * & \cdots & * & * \\ * & * & * & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & * & \cdots & * & * \\ * & * & * & * & \cdots & * & * \end{bmatrix}$$

$$\begin{array}{lll} x_1 = z_1^{u_1} z_2^{v_1} z_3^{*} \ldots z_n^{*}; & z_1 = t_1 & x_1 = t_1^{u_1} t_2^{v_1} (b_1 + c_1 t_1^{\alpha_1} + d_1 t_2^{\beta_1} + \ldots) \\ x_2 = z_1^{u_2} z_2^{v_2} z_3^{*} \ldots z_n^{*}; & z_2 = t_2 & x_2 = t_1^{u_2} t_2^{v_2} (b_2 + c_2 t_1^{\alpha_2} + d_2 t_2^{\beta_2} + \ldots) \\ x_3 = z_1^{u_3} z_2^{v_3} z_3^{*} \ldots z_n^{*}; & z_3 = b_3 + c_3 t_1^{\alpha_3} + d_3 t_2^{\beta_3} + \ldots; & x_3 = t_1^{u_3} t_2^{v_3} (b_3 + c_3 t_1^{\alpha_3} + d_3 t_2^{\beta_3} + \ldots) \\ \vdots & \vdots & \vdots \\ x_n = z_1^{u_n} z_2^{v_n} z_3^{*} \ldots z_n^{*}; & z_n = b_n + c_n t_1^{\alpha_n} + d_n t_2^{\beta_n} + \ldots; & x_n = t_1^{u_n} t_2^{v_n} (b_n + c_n t_1^{\alpha_n} + d_n t_2^{\beta_n} + \ldots) \end{array}$$

Cyclic 9-Roots Polynomial System

Two pretropisms of the cyclic 9-roots polynomial system are $U = (1, 1, -2, 1, 1, -2, 1, 1, -2)$ and $V = (0, 1, -1, 0, 1, -1, 0, 1, -1)$.
Computing initial form $In_U(C_9)(\mathbf{x})$, and then $In_V(In_U(C_9))(\mathbf{x})$ yields a system:

$$In_V(In_U(C_9))(\mathbf{x}) = \begin{cases} x_2 + x_5 + x_8 = 0 \\ x_0x_8 + x_2x_3 + x_5x_6 = 0 \\ x_0x_1x_2 + x_0x_1x_8 + x_0x_7x_8 + x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 \\ +x_4x_5x_6 + x_5x_6x_7 + x_6x_7x_8 = 0 \\ x_0x_1x_2x_8 + x_2x_3x_4x_5 + x_5x_6x_7x_8 = 0 \\ x_0x_1x_2x_3x_8 + x_0x_5x_6x_7x_8 + x_2x_3x_4x_5x_6 = 0 \\ x_0x_1x_2x_3x_4x_5 + x_0x_1x_2x_3x_4x_8 + x_0x_1x_2x_3x_7x_8 \\ +x_0x_1x_2x_6x_7x_8 + x_0x_1x_5x_6x_7x_8 + x_0x_4x_5x_6x_7x_8 + x_1x_2x_3x_4x_5x_6 \\ +x_2x_3x_4x_5x_6x_7 + x_3x_4x_5x_6x_7x_8 = 0 \\ x_0x_1x_2x_3x_4x_5x_8 + x_0x_1x_2x_5x_6x_7x_8 + x_2x_3x_4x_5x_6x_7x_8 = 0 \\ x_0x_1x_2x_3x_4x_5x_6x_8 + x_0x_1x_2x_3x_5x_6x_7x_8 + x_0x_2x_3x_4x_5x_6x_7x_8 = 0 \\ x_0x_1x_2x_3x_4x_5x_6x_7x_8 - 1 = 0 \end{cases}$$

For one of the first solutions of the cyclic 9-roots polynomial system, we refer to J. C. Faugère, *A new efficient algorithm for computing Gröbner bases (F_4).* Journal of Pure and Applied Algebra, Vol. 139, Number 1-3, Pages 61-88, Year 1999. Proceedings of MEGA'98, 22–27 June 1998, Saint-Malo, France.

Cyclic 9-Roots Polynomial System Cont.

U =(1, 1, -2, 1, 1, -2, 1, 1, -2)
V =(0, 1, -1, 0, 1, -1, 0, 1, -1)
The unimodular coordinate transformation $M : \mathbb{C}[\mathbf{x}] \rightarrow \mathbb{C}[\mathbf{z}]$ acts on the exponents. The new coordinates are given by

$$M = \begin{bmatrix} 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & -2 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} x_0 &= z_0 \\ x_1 &= z_0 z_1 \\ x_2 &= z_0^{-2} z_1^{-1} z_2 \\ x_3 &= z_0 z_3 \\ x_4 &= z_0 z_1 z_4 \\ x_5 &= z_0^{-2} z_1^{-1} z_5 \\ x_6 &= z_0 z_6 \\ x_7 &= z_0 z_1 z_7 \\ x_8 &= z_0^{-2} z_1^{-1} z_8 \end{aligned}$$

We use the coordinate change to transform the initial form system and the original cyclic 9-roots system.

Cyclic 9-Roots Polynomial System Cont.

The transformed initial form system $In_V(In_U(C_9))(\mathbf{z})$ is given by

$$\begin{cases} z_2 + z_5 + z_8 = 0 \\ z_2 z_3 + z_5 z_6 + z_8 = 0 \\ z_2 z_3 z_4 + z_3 z_4 z_5 + z_4 z_5 z_6 + z_5 z_6 z_7 + z_6 z_7 z_8 + z_2 z_3 + z_7 z_8 + z_2 + z_8 = 0 \\ z_2 z_3 z_4 z_5 + z_5 z_6 z_7 z_8 + z_2 z_8 = 0 \\ z_2 z_3 z_4 z_5 z_6 + z_5 z_6 z_7 z_8 + z_2 z_3 z_8 = 0 \\ z_2 z_3 z_4 z_5 z_6 z_7 + z_3 z_4 z_5 z_6 z_7 z_8 + z_2 z_3 z_4 z_5 z_6 + z_4 z_5 z_6 z_7 z_8 + z_2 z_3 z_4 z_5 + z_2 z_3 z_4 z_8 \\ + z_2 z_3 z_7 z_8 + z_2 z_6 z_7 z_8 + z_5 z_6 z_7 z_8 = 0 \\ z_3 z_4 z_6 z_7 + z_3 z_4 + z_6 z_7 = 0 \\ z_4 z_7 + z_4 + z_7 = 0 \\ z_2 z_3 z_4 z_5 z_6 z_7 z_8 - 1 = 0 \end{cases}$$

Its solution is
$z_2 = -\frac{1}{2} - \frac{\sqrt{3}I}{2},\ z_3 = -\frac{1}{2} + \frac{\sqrt{3}I}{2},\ z_4 = -\frac{1}{2} + \frac{\sqrt{3}I}{2},\ z_5 = 1,\ z_6 = -\frac{1}{2} - \frac{\sqrt{3}I}{2},$
$z_7 = -\frac{1}{2} - \frac{\sqrt{3}I}{2},\ z_8 = -\frac{1}{2} + \frac{\sqrt{3}I}{2}$, where $I = \sqrt{-1}$.

Cyclic 9-Roots Polynomial System Cont.

The following assignment satisfies cyclic 9-roots polynomial system **entirely**.

$$
\begin{aligned}
x_0 &= z_0 \\
x_1 &= z_0 z_1 \\
x_2 &= z_0^{-2} z_1^{-1} z_2 \\
x_3 &= z_0 z_3 \\
x_4 &= z_0 z_1 z_4 \\
x_5 &= z_0^{-2} z_1^{-1} z_5 \\
x_6 &= z_0 z_6 \\
x_7 &= z_0 z_1 z_7 \\
x_8 &= z_0^{-2} z_1^{-1} z_8
\end{aligned}
$$

$$
\begin{aligned}
z_0 &= t_1 \\
z_1 &= t_2 \\
z_2 &= -\frac{1}{2} - \frac{\sqrt{3}I}{2} \\
z_3 &= -\frac{1}{2} + \frac{\sqrt{3}I}{2} \\
z_4 &= -\frac{1}{2} + \frac{\sqrt{3}I}{2} \\
z_5 &= 1 \\
z_6 &= -\frac{1}{2} - \frac{\sqrt{3}I}{2} \\
z_7 &= -\frac{1}{2} - \frac{\sqrt{3}I}{2} \\
z_8 &= -\frac{1}{2} + \frac{\sqrt{3}I}{2}
\end{aligned}
$$

$$
\begin{aligned}
x_0 &= t_1 \\
x_1 &= t_1 t_2 \\
x_2 &= t_1^{-2} t_2^{-1} \left(-\frac{1}{2} - \frac{\sqrt{3}I}{2}\right) \\
x_3 &= t_1 \left(-\frac{1}{2} + \frac{\sqrt{3}I}{2}\right) \\
x_4 &= t_1 t_2 \left(-\frac{1}{2} + \frac{\sqrt{3}I}{2}\right) \\
x_5 &= t_1^{-2} t_2^{-1} \\
x_6 &= t_1 \left(-\frac{1}{2} - \frac{\sqrt{3}I}{2}\right) \\
x_7 &= t_1 t_2 \left(-\frac{1}{2} - \frac{\sqrt{3}I}{2}\right) \\
x_8 &= t_1^{-2} t_2^{-1} \left(-\frac{1}{2} + \frac{\sqrt{3}I}{2}\right)
\end{aligned}
$$

Recent Developments

This polyhedral method:

- very good at finding components of specific dimension

Recently, we used this polyhedral method to find solution components to all cyclic n-roots polynomial systems, where $n = m^2$, $n, m \in \mathbb{N}$.

A preprint of this result will be available on January 16^{th}.

Summary and Conjecture

What we know about dimensions of the cyclic n-roots polynomial systems, where $n = k \times m$:

$k \setminus m$	1	2	3	4	5	6	7	8
1	0	0	0	1	0	0	0	1
2	0	1	0	1	0	1		3
3	0	0	2	1				1
4	1	1	1	3		1		
5	0	0			4			
6	0	1		1		5		
7	0						6	
8	1	3	1					7

Lemma (Backelin)

If m^2 divides n, then the dimension of the cyclic n-roots polynomial system is at least $m - 1$.

Conjecture

For every cyclic n-roots polynomial system, where $n = m^2$, $n, m \in \mathbb{N}$, the largest dimension of its solution set is $m - 1$.