# Computing Puiseux Series for Algebraic Surfaces*

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**Abstract**

In this paper we outline an algorithmic approach to compute Puiseux series expansions for algebraic surfaces. The series expansions originate at the intersection of the surface with as many coordinate planes as the dimension of the surface. Our approach starts with a polyhedral method to compute cones of normal vectors to the Newton polytopes of the given polynomial system that defines the surface. If as many vectors in the cone as the dimension of the surface define an initial form system that has isolated solutions, then those vectors are potential tropisms for the initial term of the Puiseux series expansion. Our preliminary methods produce exact representations for solution sets of the cyclic $n$-roots problem, for $n = m^2$, corresponding to a result of Backelin.

**Keywords.** algebraic surface, binomial system, cyclic $n$-roots problem, initial form, Newton polytope, orbit, permutation symmetry, polyhedral method, Puiseux series, sparse polynomial system, tropism, unimodular transformation.

## 1 Introduction

We presented polyhedral algorithms to develop Puiseux expansions, for plane curves in [2] and for space curves in [1], based on ideas described in [32]. In this paper we explain a polyhedral approach to compute series developments for algebraic sets. Although we use the numerical solver of PHCpack [31], one may use any solver for the leading coefficients of the series and obtain a purely symbolic method. We implemented our methods using Sage [28].

We could reduce the treatment of algebraic sets to the curve case by adding sufficiently many hyperplanes in general position to cut out a curve on the set. This approach [26] is not flexible enough to exploit permutation symmetry as the added general hyperplanes ignore the symmetric structure of the polynomial system.

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Although presently we do not have a fully automatic implementation suitable for benchmarking on a large class of polynomial systems, we have obtained promising results on the cyclic $n$-root systems:

$$\begin{cases} x_0 + x_1 + \cdots + x_{n-1} = 0 \\ x_0 x_1 + x_1 x_2 + \cdots + x_{n-2} x_{n-1} + x_{n-1} x_0 = 0 \\ i = 3, 4, \ldots, n-1 : \displaystyle\sum_{j=0}^{n-1} \prod_{k=j}^{j+i-1} x_{k \bmod n} = 0 \\ x_0 x_1 x_2 \cdots x_{n-1} - 1 = 0. \end{cases} \tag{1}$$

The cyclic $n$-roots system is a standard benchmark problem in computer algebra, relevant to operator algebras. We refer to [30] for recent advances in the classification of complex Hadamard matrices. In [12], the close relationship of (1) with some systems occurring in optimal design of filter banks is stressed. The numerical factorization of the two dimensional surface of cyclic 9-roots into 6 irreducible cubics was reported in [25]. Recent results for the cyclic 12-roots problem can be found in [23].

Surprisingly, while looking to develop Puiseux series for algebraic sets, for cyclic 9-roots we found exact results: the first term of the series satisfies the entire polynomial system. These exact result correspond to known (see e.g. [4] or [12]) configurations of cyclic $n$-roots.

The type of polynomial systems targeted by the polyhedral approach are sparse polynomial systems. We introduce our approach in the next section with a very particular sparse class of systems. We use unimodular transformations to work with points at infinity. The second section ends with a general approach to solve a binomial system.

To find the initial coefficients in the Puiseux series we look for initial form systems, systems that have fewer monomials than the original systems and that are supported on faces of the Newton polytopes. Faces of the Newton polytopes that define the initial forms are determined by their inner normals. Those inner normals that define the initial form systems are the leading powers (called tropisms) of generalized Puiseux series. The leading coefficients of the series vanish at the initial form systems.

In the third section we define initial form systems, give an illustrative example, and describe the degeneration of a $d$-dimensional algebraic set along a path towards the intersection with the first $d$ coordinate planes. Polyhedral methods give us cones of pretropisms and initial form systems that may lead to initial coefficients of Puiseux series. We end this paper giving an exact description of positive dimensional sets of cyclic $n$-roots.

**Related work.** A geometric resolution of a polynomial system uses a parameterization of the coordinates [14] for global version of Newton's iterator [9]. Our algorithms arose from an understanding of [6, Theorem B] and are inspired by tropical methods [7] and in particular by the constructive proof of the fundamental theorem of tropical algebraic geometry [20]. Puiseux series occur perhaps most often in the resolution of singularities, [3] describes an extension of Newton's method using the notion of tropical variety. Software related to [20] is Gfan [18] and the Singular library `tropical.lib` [19].

Connections with Gröbner bases are described in [29]. Polyhedral and tropical methods for finiteness proofs in celestial mechanics are explained in [16] and [17]. Truncations of two dimensional varieties are studied in [21]. The unimodular coordinate transformations are related to power transformations in [8]. A Newton-Puiseux algorithm for polynomials in several variables is described in [5]. In [22], fractional power series solutions are developed for generic systems.

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## 2 Binomial Systems

We aim to solve *sparse* polynomial systems, systems of polynomials with relatively few monomials appearing with nonzero coefficient. The sparsest polynomial systems which admit solutions with nonzero values for all coordinates consist of *exactly two* monomials in every equation and we call such systems *binomial* systems. See e.g.: [10] and [11] for more on binomial ideals.

To represent a $d$-dimensional solution set $S$ intersecting the first $d$ coordinate planes in as many regular isolated points as the degree of $S$, the first $d$ variables can serve as independent parameters. The parameterizations that are of interest to us start with the generators of cones of normal vectors defining initial forms of polynomial systems.

### 2.1 An Example

Consider for example

$$\left\{ \begin{array}{l} x_0^2 x_1 x_2^4 x_3^3 - 1 = 0 \\ x_0 x_1 x_2 x_3 - 1 = 0. \end{array} \right. \tag{2}$$

We write the exponent vectors in the matrix

$$A = \left[ \begin{array}{cccc} 2 & 1 & 4 & 3 \\ 1 & 1 & 1 & 1 \end{array} \right] \tag{3}$$

and we look for a basis of the null space of $A$. Two linearly independent vectors that satisfy $A\mathbf{x} = \mathbf{0}$ are for example $\mathbf{u} = (-3, 2, 1, 0)$ and $\mathbf{v} = (-2, 1, 0, 1)$. Placing $\mathbf{u}$ and $\mathbf{v}$ in the columns of a matrix $M$ leads to a coordinate transformation:

$$M = \left[ \begin{array}{rrrr} -3 & -2 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \quad \left\{ \begin{array}{l} x_0 = y_0^{-3} y_1^{-2} y_2 \\ x_1 = y_0^2 y_1 y_3 \\ x_2 = y_0 \\ x_3 = y_1. \end{array} \right. \tag{4}$$

The coordinate transformation $\mathbf{x} = \mathbf{y}^M$ eliminates $y_0$ and $y_1$ — because $\mathbf{u}$ and $\mathbf{v}$ are in the null space of $A$ — as substituting the coordinates corresponds to computing $A\mathbf{u}$ and $A\mathbf{v}$, reducing the given system to

$$\left\{ \begin{array}{l} y_2^2 y_3 - 1 = 0 \\ y_2 y_3 - 1 = 0. \end{array} \right. \tag{5}$$

Solving the reduced system in (5) gives values for $y_2$ and $y_3$ which after substitution in the coordinate transformation in (4) yields an explicit solution for the original system in (2) with $y_0$ and $y_1$ as parameters.

## 2.2 Unimodular Transformations

In the previous section we constructed in (4) a unimodular coordinate transformation $\mathbf{x} = \mathbf{y}^M$, where $\det(M) = \pm 1$. In the new $\mathbf{y}$ coordinates all points that make the same inner product of the $i$th row of the given exponent matrix $A$ will have the same value for $y_i$.

The null space of the matrix $A$ is stored in the rows of the matrix $B$: $AB^T = \mathbf{0}$. The Smith normal form of $B$ consists of the triplet $(U, S, V)$, where $U$ and $V$ are unimodular matrices ($\det(U) = \pm 1$ and $\det(V) = \pm 1$), and the only nonzero elements of $S$ are on the diagonal: $UBV = S$.

If $U$ equals the identity matrix, then $UBV = S$ implies $B = SV^{-1}$. This means that for any $\mathbf{x}$, the outcome of $B\mathbf{x}$ is the same as $SV^{-1}\mathbf{x}$. If moreover $S$ contains the identity matrix, then $V^{-1}$ defines the unimodular transformation $M$. The next examples illustrates the case of general $U$ but where $S$ contains the identity matrix.

For the matrix $A$ in (3), the matrix $B$ has in its two rows the vectors $\mathbf{u}$ and $\mathbf{v}$ so that $AB^T = \mathbf{0}$:

$$B = \begin{bmatrix} -3 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{bmatrix}. \tag{6}$$

The computation of the Smith normal form of $B$ with GAP [15] (from the console in Sage [28]) gives

$$U = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{7}$$

and

$$V = \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{8}$$

We use the inverses $U^{-1}$ and $V^{-1}$ to construct a unimodular transformation extending $U^{-1}$ with the identity matrix, as follows:

$$\begin{bmatrix} -3 & 2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

and this product gives the transpose of $M$, the matrix in the unimodular transformation of (4). This examples illustrates the case when $U$ is not the identity matrix and where we may ignore $S$ as its diagonal elements are all equal to one.

We point out that the vectors in the null space of the exponent matrix $A$ as in (3) are typically normalized so that the greatest common divisors of the components of the vectors equals one. We may change coordinates so that the first vector in the null space has only its first coordinate different from zero, the second vector in the null space can have nonzero entries only in the first two coordinates, etc.

Although we prefer to represent the solution set using only integer exponents for the parameters, this is not always possible, consider for example

$$B = \left[ \begin{array}{rrrr} 2 & 6 & 17 & 9 \\ 4 & 14 & 13 & 3 \end{array} \right]. \tag{10}$$

The divisors for the two rows of $B$ (and the denominators of the exponents of the parameters) are obtained via the Hermite normal form of $B$: $UB = H$, where $U$ is a square unimodular matrix and $H$ an upper triangular matrix. We assume that $B$ is full rank and that the columns have been permuted so $H$ has only nonzero elements on its diagonal. Let $D$ be a diagonal matrix of the same dimensions as $U$ which takes its elements from the corresponding diagonal elements of the matrix $H$. Then the coordinate transformation is defined by

$$M = \left[ \begin{array}{cc} \multicolumn{2}{c}{D^{-1}B} \\ \mathbf{0} & I \end{array} \right]. \tag{11}$$

where $I$ is the identity matrix. To show that the determinant of $M$ equals $\pm 1$, consider the extended unimodular matrix

$$\widehat{U} = \left[ \begin{array}{cc} U & \mathbf{0} \\ \mathbf{0} & I \end{array} \right]. \tag{12}$$

Because $U$ is unimodular, $\det(\widehat{U}) = \pm 1$ and $\det(\widehat{U}M) = \pm \det(M)$. We have $\det(\widehat{U}M) = \pm 1$, because $\widehat{U}M$ is an upper triangular matrix with $\pm 1$ on its diagonal as a result of the multiplication by $D^{-1}$.

Note that the rational exponents will appear only in the powers of the parameters as performing the coordinate transformation $\mathbf{x} = \mathbf{y}^M$ on the system $\mathbf{x}^A - \mathbf{c} = \mathbf{0}$ eliminates the first $d$ variables of the $d$-dimensional solution set.

## 2.3 Solving Binomial Systems

We denote a binomial system by $\mathbf{x}^A - \mathbf{c} = \mathbf{0}$, where $A \in \mathbb{Z}^{k \times n}$ and $\mathbf{c} = (c_0, c_1, \ldots, c_{k-1})^T$ with $c_i \neq 0$ for all $i = 0, 1, \ldots, k-1$. If the rank of $A$ equals $k$, then $k$ is the codimension of the solution set. Given the tuple $(A, \mathbf{c})$, the solution set of $\mathbf{x}^A - \mathbf{c} = \mathbf{0}$ is described by a unimodular transformation $M$ and a set of values for the last $n - k$ variables.

In the sketch of the solution method below we assume that $A$ has rank $k$, otherwise $\mathbf{x}^A - \mathbf{c} = \mathbf{0}$ has no $(n-k)$-dimensional solution set for general values of $\mathbf{c}$. The steps are as follows:

1. Compute the null space $B$ of $A$, $d = n - k$.
2. Compute the Smith normal form $(U, S, V)$ of $B$.
3. Depending on $U$ and $S$ do one of the following:
   - If $U$ is the identity matrix, then $M = V^{-1}$ and the first $d$ variables have positive denominators in their powers when not all elements on the diagonal of $S$ are equal to one.

- If $U$ is not the identity matrix and if all elements on the diagonal of $S$ are one, then extend $U^{-1}$ with an identity matrix to obtain an $n$-by-$n$ matrix $E$ that has $U^{-1}$ in its first $d$ rows and columns. Then, $M = EV^{-1}$.
- In all other cases, define $M$ as in (11).

4. After the coordinate transformation $\mathbf{x} = \mathbf{y}^M$, compute the leading coefficients solving a binomial system of $k$ equations in $k$ unknowns. Return $M$ and the corresponding solutions of the binomial system.

The solution procedure for binomial systems outlined above returns a representation with $d$ parameters for the $d$-dimensional solution set which can geometrically interpreted as follows. For zero values of the parameters, we obtain the points of the solution set intersected with the first $d$ coordinate hyperplanes. For nonzero values of the parameters, the powers of the parameters correspond to a choice of the basis for the null space of the exponent matrix of the binomial system.

# 3 Sparse Polynomial Systems

To look for $d$-dimensional components of sparse polynomial systems, we investigate solutions of initial forms defined by cones of normal vectors. In order for the initial form systems to have solutions with all coordinates different from zero, they need to be at least binomial systems.

Although not all (and perhaps only few) initial form systems are binomial, the unimodular transformations explained in §2.2 are applied on a matrix of pretropisms.

## 3.1 Initial Forms

A polynomial $f$ in $n$ variables $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ is denoted as

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad c_{\mathbf{a}} \in \mathbb{C} \setminus \{0\}, \tag{13}$$

$x^{\mathbf{a}} = x_0^{a_0} x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}$, where $A$ is the set of all exponents of monomials with nonzero coefficient. The set $A$ is the *support* of $f$ and the convex hull of $A$ is the *Newton polytope* $P$ of $f$. Any nonzero vector $\mathbf{v}$ defines a face of $P$, spanned by

$$\mathrm{in}_{\mathbf{v}}(A) = \{ \ \mathbf{b} \in A \mid \langle \mathbf{b}, \mathbf{v} \rangle = \min_{\mathbf{a} \in A} \langle \mathbf{a}, \mathbf{v} \rangle \ \}, \tag{14}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product of two vectors. We use the notation $\mathrm{in}_{\mathbf{v}}(A)$ because a face of a support set defines an *initial form* of the polynomial $f$:

$$\mathrm{in}_{\mathbf{v}}(f)(\mathbf{x}) = \sum_{\mathbf{a} \in \mathrm{in}_{\mathbf{v}}(A)} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \tag{15}$$

where $A$ is the support of $f$. For a system $f(\mathbf{x}) = \mathbf{0}$ and a nonzero vector $\mathbf{v}$, the *initial form system* $\mathrm{in}_{\mathbf{v}}(f)(\mathbf{x}) = \mathbf{0}$ is defined by the initial forms of the polynomials $f$ with respect to $\mathbf{v}$.

Because the initial coefficients of Puiseux series expansions are solutions to initial form systems, the initial forms we consider must have at least two monomials, otherwise the solutions will have coordinates equal to zero and are unfit as leading coefficients in a Puiseux series development.

### 3.2 An Illustrative Example

In this section we indicate how the presence of a higher dimensional solution set manifests itself from the relative position of the Newton polytopes of the polynomials in the system. To illustrate a numerical irreducible decomposition of the solution set of a polynomial system, the following system was used in [24]:

$$f(x,y,z) = \left\{ \begin{array}{r} (y-x^2)(x^2+y^2+z^2-1)(x-0.5) = 0 \\ (z-x^3)(x^2+y^2+z^2-1)(y-0.5) = 0 \\ (y-x^2)(z-x^3)(x^2+y^2+z^2-1)(z-0.5)=0 \end{array} \right. \tag{16}$$

The solution set $Z = f^{-1}(\mathbf{0})$ is decomposed as

$$\begin{aligned} Z &= Z_2 \cup Z_1 \cup Z_0 && (17) \\ &= \{Z_{21}\} \cup \{Z_{11} \cup Z_{12} \cup Z_{13} \cup Z_{14}\} \cup \{Z_{01}\} && (18) \end{aligned}$$

where

1. $Z_{21}$ is the sphere $x^2 + y^2 + z^2 - 1 = 0$,
2. $Z_{11}$ is the line $(x = 0.5, z = 0.5^3)$,
3. $Z_{12}$ is the line $(x = \sqrt{0.5}, y = 0.5)$,
4. $Z_{13}$ is the line $(x = -\sqrt{0.5}, y = 0.5)$,
5. $Z_{14}$ is the twisted cubic $(y - x^2 = 0, z - x^3 = 0)$,
6. $Z_{01}$ is the point $(x = 0.5, y = 0.5, z = 0.5)$.

A first cascade of homotopies in [24] needed 197 solution paths to compute generic points on all components. The equation-by-equation solver of [27] reduced the number of paths down to 13. The Newton polytopes of the polynomials in the system are displayed in Figures 1 and 2.

Consider a point on the 2-dimensional solution component of $f^{-1}(\mathbf{0})$ and let the first coordinate of that point go to zero. As $x_1 = t \to 0$:

$$\mathrm{in}_{(1,0,0)}(f)(x,y,z) = \left\{ \begin{array}{r} y(y^2+z^2-1)(-0.5) = 0 \\ z(y^2+z^2-1)(y-0.5) = 0 \\ yz(y^2+z^2-1)(z-0.5) = 0. \end{array} \right. \tag{19}$$

Alternatively, as $x_2 = s \to 0$, we end up at a solution of the initial form system:

$$\mathrm{in}_{(0,1,0)}(f)(x,y,z) = \left\{ \begin{array}{r} -x^2(x^2+z^2-1)(x-0.5) = 0 \\ (z-x^3)(x^2+z^2-1)(-0.5) = 0 \\ -x^2(z-x^3)(x^2+z^2-1)(z-0.5) = 0. \end{array} \right. \tag{20}$$

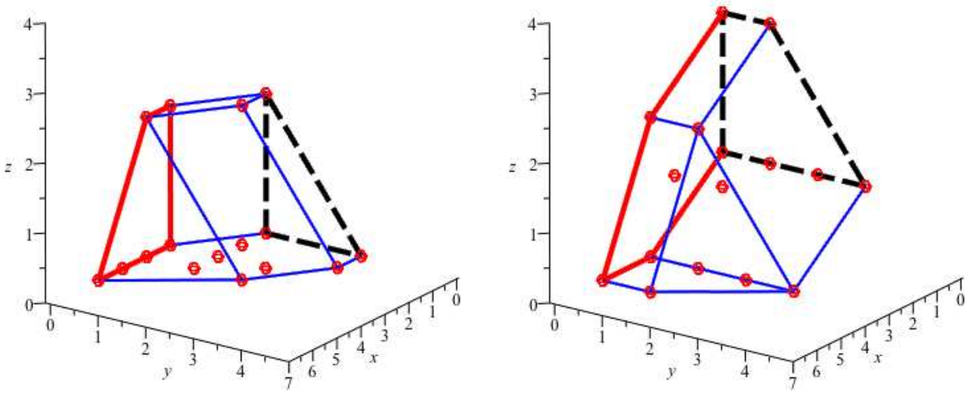

Figure 1: From top to bottom, we see the Newton polytopes of $f_1$ and $f_2$, the polynomials in (16). The edges of the faces of the polytopes with normals $(1,0,0)$ and $(0,1,0)$ are marked in bold, respectively in red (thick solid lines) and black (thick dashed lines).

Looking at the Newton polytopes along $\mathbf{v} = (1,0,0)$ and $\mathbf{v} = (0,1,0)$, we consider faces of the Newton polytopes, see Figures 1 and 2.

Combining the two degenerations, we arrive at the initial form system:

$$\begin{aligned} &\mathrm{in}_{(0,1,0)}(\mathrm{in}_{(1,0,0)}(f))(x,y,z) \\ &= \begin{cases} y(z^2-1)(-0.5) = 0 \\ z(z^2-1)(-0.5) = 0 \\ yz(z^2-1)(z-0.5) = 0 \end{cases} \end{aligned} \tag{21}$$

The factor $z^2 - 1$ is shared with $\mathrm{in}_{(1,0,0)}(\mathrm{in}_{(0,1,0)}(f))(x,y,z)$.

Based on these degenerations, we arrive at the following representation for a solution surface. The sphere is two dimensional, $x$ and $y$ are free:

$$\begin{cases} x = t_0 \\ y = t_1 \\ z = 1 + c_0 t_0^2 + c_1 t_1^2. \end{cases} \tag{22}$$

For $t_0 = 0$ and $t_1 = 0$, $z = 1$ is a solution of $z^2 - 1 = 0$. Substituting $(x = t_0, y = t_1, z = 1 + c_0 t_0^2 + c_1 t_1^2)$ into the original system gives linear conditions on the coefficients of the second term: $c_0 = -0.5$ and $c_1 = -0.5$.

### 3.3 Asymptotics of Algebraic Surfaces and Puiseux Series

Denoting by $d$ the dimension of the algebraic surface defined by $f(\mathbf{x}) = \mathbf{0}$, for $\mathbf{x} \in \mathbb{C}^n$, we assume the defining equations are in Noether position so we may specialize the first $d$ coordinates to

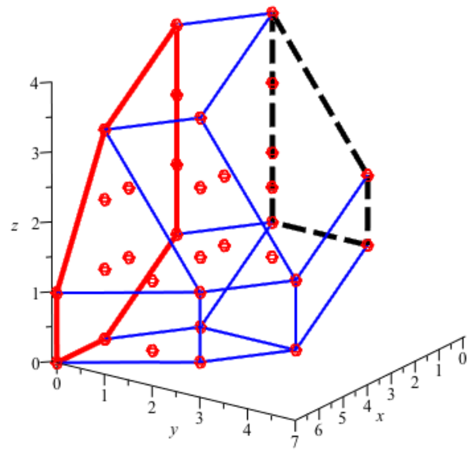


Figure 2: The Newton polytopes of the third polynomial in (16). The edges of the faces of the polytopes with normals $(1,0,0)$ and $(0,1,0)$ are marked in bold, respectively in red (thick solid lines) and black (thick dashed lines).

random complex numbers in $f(\mathbf{x}) = \mathbf{0}$ and obtain a system with isolated solutions. Moreover, we assume that when specializing the first $d$ variables to zero, the algebraic set remains of dimension $d$. Geometrically this means that we assume that the algebraic set meets the first $d$ coordinate planes (perpendicular to the first $d$ coordinate axes) properly.

We consider what happens when starting at a random point on the surface we move the first $d$ coordinates to zero. For simplicity of notation we take $d = 2$ and consider a multiparameter family of polynomial systems:

$$\left\{ \begin{array}{rcl} f(\mathbf{x}) & = & \mathbf{0} \\ x_0 & = & c_0 t_0 \\ x_1 & = & c_1 t_0^{v_{0,1}} t_1^{v_{1,1}} (c_{1,1} + O(t_0, t_1)), \end{array} \right. \tag{23}$$

with $c_0, c_1, c_{1,1} \in \mathbb{C} \setminus \{0\}$, $v_{0,1}, v_{1,1} \in \mathbb{Q}$, letting $t_0$ and $t_1$ go from 1 to 0, starting at a generic point on the surface with its first two coordinates equal to $c_0$ and $c_1$.

The multiparameter family in (23) specifies the last equation as a series to leave enough freedom for the actual shape of the surface. While we may always move $x_0$ as going linearly to zero, with $x_0 = c_0 t_0$, the second coordinate of a point along a path on the surface may no longer move linearly. Taking $x_1$ as $c_1 t_1$ would be too restrictive.

As we move $x_0$ to zero as $t_0$ goes to zero, then $x_1$ can go to zero as well if $v_{0,1} > 0$ and $v_{1,1} > 0$, or go to infinity if $v_{0,1} < 0$ or $v_{1,1} < 0$, or go to $c_1 c_{1,1}$ if both $v_{0,1} = 0$ and $v_{1,1} = 0$. The multiparameter family in (23) contains what we define as a multiparameter version of a Puiseux series for algebraic curves. Similar to $x_1$, the other components of the moving point can be developed as a generalized Puiseux series

$$x_k = c_k t_0^{v_{0,k}} t_1^{v_{1,k}} (c_{1,k} + O(t_0, t_1)), \tag{24}$$

$c_k, c_{1,k} \in \mathbb{C} \setminus \{0\}$, $v_{0,k}, v_{1,k} \in \mathbb{Q}$. If in the limit — when $t_0$ and $t_1$ are both zero — the solution is finite and of multiplicity one, and if the powers in the series are positive integer numbers, then the generalized Puiseux series coincides with a multivariate Taylor series.

As $t_0$ and $t_1$ go to zero, the system $F(t_0, t_1) = \mathbf{0}$ — obtained after replacing $x_0$ and $x_1$ using the last two equations of (23) and after substituting (24) for the remaining $n-2$ into $f(\mathbf{x}) = \mathbf{0}$ — must have at least two monomials with lowest power in $t_0$ and lowest power in $t_1$ in every equation because $c_k, c_{1,k} \in \mathbb{C} \setminus \{0\}$ for all $k = 0, 1, \ldots, n-1$. We call the part of $f(\mathbf{x}) = \mathbf{0}$ corresponding to $F(t_0, t_1)$ with lowest powers of $t_0$ and $t_1$ *the initial form system* of $f(\mathbf{x}) = \mathbf{0}$ with respect to the normal vectors $\mathbf{v}_0 = (1, v_{0,1}, v_{0,1}, \ldots, v_{0,n-1})$ and $\mathbf{v}_1 = (0, v_{1,1}, v_{1,2}, \ldots, v_{1,n-1})$. Because the normal vectors are the leading powers of the generalized Puiseux series, $\mathbf{v}_0$ and $\mathbf{v}_1$ can be called *tropisms* in analogy to the case of algebraic curves.

The derivation of Puiseux series for an algebraic set in any dimension $d$ if formulated as follows.

**Proposition 3.1.** *If $f(\mathbf{x}) = \mathbf{0}$ is in Noether position and defines a $d$-dimensional solution set in $\mathbb{C}^n$, intersecting the first $d$ coordinate planes in regular isolated points, then there are $d$ linearly independent tropisms $\mathbf{v}_0, \mathbf{v}_1, \ldots \mathbf{v}_{d-1} \in \mathbb{Q}^n$ so that the initial form system $\mathrm{in}_{\mathbf{v}_0}(\mathrm{in}_{\mathbf{v}_1}(\cdots \mathrm{in}_{\mathbf{v}_{d-1}}(f)\cdots))(\mathbf{x} = \mathbf{y}^M) = \mathbf{0}$ has a solution $\mathbf{c} \in (\mathbb{C} \setminus \{0\})^{n-d}$. This solution and the tropisms are the leading coefficients and powers of a generalized Puiseux series expansion for the algebraic set:*

$$\begin{array}{rcl} x_0 & = & t_0^{v_{0,0}} \\ x_1 & = & t_0^{v_{0,1}} t_1^{v_{1,1}} \\ & \vdots & \\ x_{d-1} & = & t_0^{v_{0,d-1}} t_1^{v_{1,d-1}} \cdots t_{d-1}^{v_{d-1,d-1}} \\ x_d & = & c_0 t_0^{v_{0,d}} t_1^{v_{1,d}} \cdots t_{d-1}^{v_{d-1,d}} + \cdots \\ x_{d+1} & = & c_1 t_0^{v_{0,d+1}} t_1^{v_{1,d+1}} \cdots t_{d-1}^{v_{d-1,d+1}} + \cdots \\ & \vdots & \\ x_n & = & c_{n-d-1} t_0^{v_{0,n-1}} t_1^{v_{1,n-1}} \cdots t_{d-1}^{v_{d-1,n-1}} + \cdots \end{array} \quad (25)$$

*Proof.* Because the set defined by $f(\mathbf{x}) = \mathbf{0}$ is in Noether position, we can let the first $d$ variables go to zero, using for example a multiparameter homotopy as in (23) and still obtain regular isolated solutions, denoted as $(0, 0, \ldots, 0, c_0, c_1, \ldots, c_{n-d-1}) \in \mathbb{C}^n$.

The tropisms $\mathbf{v}_0$, $\mathbf{v}_1$, …, $\mathbf{v}_{d-1}$ define the initial form system, i.e.: those monomials in the system $f(\mathbf{x}) = \mathbf{0}$ that become dominant as the parameters $t_0$, $t_1$, …, $t_{d-1}$ move to zero. In particular: for any vector $\mathbf{v}$ in the cone spanned by the tropisms, we have that every monomial $\mathbf{x}^{\mathbf{a}}$ in the initial form system makes minimal inner product $\langle \mathbf{a}, \mathbf{v} \rangle$, minimal with respect to any other monomial $\mathbf{x}^{\mathbf{b}}$ not in the initial form system, i.e.: $\langle \mathbf{a}, \mathbf{v} \rangle$ ¡ $\langle \mathbf{b}, \mathbf{v} \rangle$.

Because the leading terms of the Puiseux series vanish at the initial form system, the inner product with the monomials and the leading powers must be minimal compared to all other monomials in the system. Hence the shape of the Puiseux series. □

### 3.4 Polyhedral Methods

In our algorithm to develop Puiseux series developments for algebraic sets, Proposition 3.1 is applied as follows. If we are looking for an algebraic set of dimension $d$ and

- if there are no cones of vectors perpendicular to edges of the Newton polytopes of $f(\mathbf{x}) = \mathbf{0}$ of dimension $d$, then the system $f(\mathbf{x}) = \mathbf{0}$ has no solution set of dimension $d$ that intersects the first $d$ coordinate planes properly; otherwise

- if a $d$-dimensional cone of vectors perpendicular to edges of the Newton polytopes exists, then that cone defines a part of the tropical prevariety.

We call a vector perpendicular to at least one edge of every Newton polytope of $f(\mathbf{x}) = \mathbf{0}$ a candidate tropism or *pretropism*.

Algorithms to compute a tropical prevariety are described in [7]. As we outlined in [1], we applied `cddlib` [13] to the Cayley embedding of the Newton polytopes of the system to compute pretropisms. With the Cayley embedding we managed to compute all pretropisms of the cyclic 12-roots problem, reported in [1].

For highly structured problems such as the cyclic $n$-roots problem, a tropism found at lower dimension often occurs also in extended form for higher dimensions. For example, for $n = 4$, a tropism is $(+1, -1, +1, -1)$ which extends directly to $(+1, -1, +1, -1, +1, -1, +1, -1)$ for $n = 8$ and $(+1, -1, +1, -1, +1, -1, +1, -1, +1, -1, +1, -1)$ for $n = 12$, and any $n$ that is a multiple of 4.

In addition to the extraneous results reported from the Cayley embedding, it suffices to restrict to pretropisms with positive first coordinate because geometrically we intersect the solution set with the coordinate hyperplane perpendicular to the $x_0$-axes at the end of moving $x_0$ to zero. Allowing a negative first exponent in the first pretropism corresponds to intersecting the solution set at infinity, when in the limit we let $x_0$ go to infinity.

In any case, after the computation of pretropisms, exploiting permutation symmetry is relatively straightforward as we can group the pretropisms in orbits and process only one generator per orbit.

### 3.5 Puiseux Series for Algebraic Sets

The approach to develop Puiseux series proceeds as follows. For every $d$-dimensional cone $C$ of pretropisms:

1. We select $d$ linearly independent generators to form the $d$-by-$n$ matrix $A$ and the corresponding unimodular transformation $\mathbf{x} = \mathbf{y}^M$.

2. Because the matrix $A$ contains pretropisms, the initial form system $\text{in}_{\mathbf{v}_0}(\text{in}_{\mathbf{v}_1}(\cdots \text{in}_{\mathbf{v}_{d-1}}(f) \cdots))(\mathbf{x}) = \mathbf{0}$ determined by the rows $\mathbf{v}_0$, $\mathbf{v}_1$, …, $\mathbf{v}_{d-1}$ of $A$ has at least two monomials in every equation. If the initial form system has no solution with all coordinates different from zero, then we move to the next cone $C$ and return to step 1, else we continue with the next step.

3. Solutions of the initial form system found in the previous step may be leading coefficients in a potential Puiseux series with corresponding leading powers equal to the pretropisms. If the leading term satisfies the entire polynomial system, then we report an explicit solution of the system and we continue processing the next cone $C$. Otherwise, we take the current leading term to the next step.

4. If there is a second term in the Puiseux series, then we have computed an initial development for an algebraic set and report this development in the output.

To compute in the last step a second term in a multivariate Puiseux series seems very complicated, but we point out that it is not necessary to compute the second term in all $d$ variables. To ensure that a solution of an initial form system is not isolated, it suffices that we can compute a series development *for a curve* starting at that solution. In practice this means that we may restrict all but one free variable in the series development and apply the methods we outlined in [1] for the computation of the second term of the Puiseux series for a space curve.

With Puiseux series, the solutions of the initial form system can be extended to form a witness set. A witness set [26] is a numerical data structure for positive dimensional solution sets of polynomial systems. Depending on the heights of the powers in the series, we may need more than the second term to ensure convergence with Newton's method.

# 4 Applications

Our polyhedral approach enables to compute exact representations for positive dimensional solution sets of the cyclic $n$-roots problem (1).

## 4.1 On cyclic 9-roots

Taking $n = 9$ in (1), for cyclic 9-roots, we show that our solution can be transformed into the same format as in the proof we found in [12, Lemma 1.1] of the statement in [4] that square divisors of $n$ lead to infinitely many cyclic $n$-roots.

Among the tropisms computed by `cddlib` [13] on the Cayley embedding of the Newton polytopes of the system, there is a two dimensional cone of normal vectors spanned by $\mathbf{u} = (1, 1, -2, 1, 1, -2, 1, 1, -2)$ and $\mathbf{v} = (0, 1, -1, 0, 1, -1, 0, 1, -1)$. The vectors $\mathbf{u}$ and $\mathbf{v}$ are tropisms. The initial form system $\text{in}_{\mathbf{u}}(\text{in}_{\mathbf{v}}(f))(\mathbf{x}) = \mathbf{0}$ is

$$
\left\{
\begin{array}{rcl}
x_2 + x_5 + x_8 & = & 0 \\
x_0x_8 + x_2x_3 + x_5x_6 & = & 0 \\
x_0x_1x_2 + x_0x_1x_8 + x_0x_7x_8 + x_1x_2x_3 & & \\
+ \; x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_6 + x_5x_6x_7 & & \\
+ \; x_6x_7x_8 & = & 0 \\
x_0x_1x_2x_8 + x_2x_3x_4x_5 + x_5x_6x_7x_8 & = & 0 \\
x_0x_1x_2x_3x_8 + x_0x_5x_6x_7x_8 + x_2x_3x_4x_5x_6 & = & 0 \\
x_0x_1x_2x_3x_4x_5 + x_0x_1x_2x_3x_4x_8 & & \\
+ \; x_0x_1x_2x_3x_7x_8 + x_0x_1x_2x_6x_7x_8 & & \\
+ \; x_0x_1x_5x_6x_7x_8 + x_0x_4x_5x_6x_7x_8 & & \\
+ \; x_1x_2x_3x_4x_5x_6 + x_2x_3x_4x_5x_6x_7 & & \\
+ \; x_3x_4x_5x_6x_7x_8 & = & 0 \\
x_0x_1x_2x_3x_4x_5x_8 + x_0x_1x_2x_5x_6x_7x_8 & & \\
+ \; x_2x_3x_4x_5x_6x_7x_8 & = & 0 \\
x_0x_1x_2x_3x_4x_5x_6x_8 + x_0x_1x_2x_3x_5x_6x_7x_8 & & \\
+ \; x_0x_2x_3x_4x_5x_6x_7x_8 & = & 0 \\
x_0x_1x_2x_3x_4x_5x_6x_7x_8 - 1 & = & 0.
\end{array}
\right.
\tag{26}
$$

Although not binomial, $\text{in}_{\mathbf{u}}(\text{in}_{\mathbf{v}}(f))(\mathbf{x}) = \mathbf{0}$ is is significantly sparser and thus easier to solve than the original system. To solve $\text{in}_{\mathbf{u}}(\text{in}_{\mathbf{v}}(f))(\mathbf{x}) = \mathbf{0}$, we eliminate $x_0$ and $x_1$ with a unimodular coordinate transformation $M$ that has $\mathbf{u}$ and $\mathbf{v}$ on its first two rows. The last seven rows of $M$ are zero except for the ones on the diagonal:

$$
M = \left[
\begin{array}{rrrrrrrrr}
1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & -2 \\
0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
\right].
\tag{27}
$$

The matrix $M$ defines the unimodular coordinate transformation $\mathbf{x} = \mathbf{y}^M$:

$$
\begin{array}{lll}
x_0 = y_0 & x_3 = y_0y_3 & x_6 = y_0y_6 \\
x_1 = y_0y_1 & x_4 = y_0y_1y_4 & x_7 = y_0y_1y_7 \\
x_2 = y_0^{-2}y_1^{-1}y_2 & x_5 = y_0^{-2}y_1^{-1}y_5 & x_8 = y_0^{-2}y_1^{-1}y_8.
\end{array}
\tag{28}
$$

The transformation $\mathbf{x} = \mathbf{y}^M$ reduces the initial form system $\text{in}_{\mathbf{u}}(\text{in}_{\mathbf{v}}(f))(\mathbf{x} = \mathbf{y}^M) = \mathbf{0}$ to a system of 9 equations in 7 unknowns.

After adding two slack variables to square the system (see [26] for an illustration of introducing slack variables), the mixed volume equals 326. In contrast, the mixed volume of the original polynomial system equals 20,376.

We find that the entire cyclic 9-roots system vanishes at this first term of the series expansion. Recognizing the numerical roots as primitive roots of unity leads to an exact representation of the two dimensional set of cyclic 9-roots.

Denoting by $u = e^{i2\pi/3}$ the primitive third root of unity, $u^3 - 1 = 0$, our representation of the solution set is

$$\begin{array}{lll} x_0 = t_0 & x_3 = t_0 u & x_6 = t_0 u^2 \\ x_1 = t_0 t_1 & x_4 = t_0 t_1 u & x_7 = t_0 t_1 u^2 \\ x_2 = t_0^{-2} t_1^{-1} u^2 & x_5 = t_0^{-2} t_1^{-1} & x_8 = t_0^{-2} t_1^{-1} u. \end{array} \quad (29)$$

Introducing new variables $y_0 = t_0$, $y_1 = t_0 t_1$, and $y_2 = t_0^{-2} t_1^{-1} u^2$, our representation becomes

$$\begin{array}{lll} x_0 = y_0 & x_3 = y_0 u & x_6 = y_0 u^2 \\ x_1 = y_1 & x_4 = y_1 u & x_7 = y_1 u^2 \\ x_2 = y_2 & x_5 = y_2 u & x_8 = y_2 u^2 \end{array} \quad (30)$$

which modulo $y_0^3 y_1^3 y_2^3 u^9 - 1 = 0$ satisfies by plain substitution the cyclic 9-roots system, as in the proof of [12, Lemma 1.1].

Note that the representation in (29) allows a quick computation of the degree of the surface. This degree equals the number of points in the intersection of the surface with two random hyperplanes. Using (29) for points on the surface, the two random hyperplanes become a system in the monomials $t_0$, $t_0 t_1$, and $t_0^{-2} t_1^{-1}$:

$$\left\{ \begin{array}{l} \alpha_1 t_0 + \alpha_{1,2} t_0 t_1 + \alpha_{-2,-1} t_0^{-2} t_1^{-1} = 0 \\ \beta_1 t_0 + \beta_{1,2} t_0 t_1 + \beta_{-2,-1} t_0^{-2} t_1^{-1} = 0 \end{array} \right. \quad (31)$$

for some complex numbers $\alpha_{i,j}$ and $\beta_{i,j}$. The above system is equivalent to the system

$$\left\{ \begin{array}{r} t_0^{-3} t_1^{-1} - c_0 = 0 \\ t_1 - c_1 = 0 \end{array} \right. \quad (32)$$

for some $c_0, c_1 \in \mathbb{C}$. We see that for any nonzero $c_0$ and $c_1$, the system has three solutions. So the algebraic surface represented in (29) is a cubic surface. Using other roots of unity and permuting variables leads to an entire orbit of cubic surfaces.

Using the representation (30), we arrange the position of the coefficients with $u$ as a third root of unity:

$$\begin{array}{ccc} 1 & u & u^2 \\ u & u^2 & 1 \\ u^2 & 1 & u \\ u^2 & u & 1 \\ u & 1 & u^2 \\ 1 & u^2 & u \end{array} \quad (33)$$

shifting the variables in forward and backward order. So the one cubic surface leads to an orbit of 6 cubic surfaces, corresponding with our numerical results of [25].

## 4.2 On cyclic $m^2$-roots

While the Cayley embedding becomes too wasteful to extend the computation of *all* candidate tropisms beyond $n = 12$, by the structure of the tropisms for $n = 9$ we can predict the tropisms for cyclic 16-roots:

$$\begin{aligned}\mathbf{u} &= (1,1,1,-3,1,1,1,-3,1,1,1,-3,1,1,1,-3),\\ \mathbf{v} &= (0,1,1,-2,0,1,1,-2,0,1,1,-2,0,1,1,-2),\\ \mathbf{w} &= (0,0,1,-1,0,0,1,-1,0,0,1,-1,0,0,1,-1),\end{aligned} \tag{34}$$

and the corresponding initial form solutions are primitive fourth roots of unity. Similar to (29) and (30) we can show that the exact representation obtained with tropical methods corresponds to what is in the proof of [12, Lemma 1.1].

A general pattern for surfaces of cyclic $m^2$-roots is below.

**Proposition 4.1.** *For $n = m^2$, there is an $(m-1)$-dimensional set of cyclic $n$-roots, represented exactly as*

$$\begin{array}{rcl} x_{km+0} & = & u_k t_0 \\ x_{km+1} & = & u_k t_0 t_1 \\ x_{km+2} & = & u_k t_0 t_1 t_2 \\ & \vdots & \\ x_{km+m-2} & = & u_k t_0 t_1 t_2 \cdots t_{m-2} \\ x_{km+m-1} & = & u_k t_0^{-m+1} t_1^{-m+2} \cdots t_{m-3}^{-2} t_{m-2}^{-1} \end{array} \tag{35}$$

*for $k = 0,1,2,\ldots,m-1$ and $u_k = e^{i2k\pi/m}$.*

The substitution $t_0 = s_0$, $t_0t_1 = s_1$, $t_0t_1t_2 = s_2$, …, $t_0^{-m+1}t_1^{-m+2}\cdots t_{m-3}^{-2}t_{m-2}^{-1} = s_0^{-1}s_1^{-1}\cdots s_{m-2}^{-1}$ simplifies (35).

**Proposition 4.2.** *The $(m-1)$-dimensional solution set in (35) has degree equal to $m$.*

*Proof.* To determine the degree of an $(m-1)$-dimensional algebraic set, we intersect the set with $m-1$ hyperplanes with random coefficients. In any linear equation we replace the $x$-variables using the equations in (35), dividing each equation by $t_0$ to obtain a nonzero constant coefficient. Because every $x_j$ corresponds to one monomial in $t_0$, $t_1$, …, $t_{m-1}$, bringing the coefficient matrix into a reduced row echelon form leads to a binomial system of $m-1$ equations in $m-1$ unknowns:

$$\left\{ \begin{array}{rcl} t_0^{-m}t_1^{-m+2}t_2^{-m+3}\cdots t_{m-2}^{-1} - c_0 & = & 0 \\ t_1 - c_1 & = & 0 \\ t_1t_2 - c_2 & = & 0 \\ & \vdots & \\ t_1t_2t_3\cdots t_{m-2} - c_{m-2} & = & 0 \end{array} \right. \tag{36}$$

Collecting the coefficients $(c_0, c_1, c_2, \ldots, c_{m-2})$ in $\mathbf{c}$ and the exponents in a matrix $A$, we denote the binomial system as $\mathbf{t}^A = \mathbf{c}$ with

$$A = \begin{bmatrix} -m & -m+2 & -m+3 & \cdots & -2 & -1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}. \tag{37}$$

The binomial system has $|\det(A)| = m$ solutions and therefore the degree equals $m$. □

Applying the permutation symmetry, shifting the variables forward and backward as in (33), we find $2m$ components of degree $m$.

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