Solving Polynomial Systems in Noether Position with Puiseux Series

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Talk Outline

a new polyhedral method

- development of positive dimensional solution sets
- for square systems and systems with more equations than unknowns
  - i.e. systems where positive dimensional solution sets are not expected

in applications

- cyclic n-roots problem

main result

- tropical version of Backelin's Lemma

Polyhedral Method Description

origins of the method

- polyhedral homotopies and the work of Bernshtein
- Bernshtein's Theorems A & B
- solve systems with Puiseux series and Bernshtein's Theorems A & B

main aim

- to generalize polyhedral homotopies
- zero-dimensional solution sets $\rightarrow$ general algebraic sets

we are inspired in part by the constructive proof of

Theorem (Fundamental Theorem of Tropical Algebraic Geometry)

$\omega \in Trop(I) \cap \mathbb{Q}^n \iff \exists p \in V(I) : -val(p) = \omega \in \mathbb{Q}^n.$

A.N. Jensen, H. Markwig, T. Markwig. *An Algorithm for Lifting Points in a Tropical Variety*. Collect. Math. vol. 59, no. 2, pages 129–165, 2008.

rephrasing the theorem

rational vector in the tropical variety corresponds to the leading powers of a Puiseux series, converging to a point in the algebraic variety.

- we understand the fundamental theorem via polyhedral homotopies
- we see it as a generalization of Bernshtein's Theorem B

General Definitions

Definition (Polynomial System)

$$F(\mathbf{x}) = \begin{cases} f_0(\mathbf{x}) = 0 \\ f_1(\mathbf{x}) = 0 \\ \vdots \\ f_{n-1}(\mathbf{x}) = 0 \end{cases}$$

Definition (Laurent Polynomial)

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad c_{\mathbf{a}} \in \mathbb{C} \setminus \{0\}, \quad \mathbf{x}^{\mathbf{a}} = x_0^{\pm a_0} x_1^{\pm a_1} \cdots x_{n-1}^{\pm a_{n-1}}$$

Definition (Support Set)

The set of exponents A_i is called the **support set** of f_i.

Definition (Newton Polytope)

Let A_i be the support set of the polynomial $f_i \in \mathbf{F}(\mathbf{x}) = \mathbf{0}$. Then, the **Newton polytope** of f_i is the convex hull of A_i, denoted P_i.

General Definitions

Definition (Initial Form)

Let $f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ be a Laurent polynomial, $\mathbf{v} \in \mathbb{Z}^n$ a non-zero vector and let $\langle \cdot, \cdot \rangle$ denote the usual inner product. Then, the **initial form** with respect to $\mathbf{v}$ is given by

$$in_{\mathbf{v}}(f(\mathbf{x})) = \sum_{\mathbf{a} \in A,\ m = \langle \mathbf{a}, \mathbf{v} \rangle} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \qquad m = min\{\langle \mathbf{a}, \mathbf{v} \rangle \mid \mathbf{a} \in A\}$$

where the minimal value m has been achieved at least twice.

Definition (Initial Form System)

For a system of polynomials $\mathbf{F}(\mathbf{x}) = \mathbf{0}$, the **initial form system** is defined by $in_{\mathbf{v}}(\mathbf{F}(\mathbf{x})) = (in_{\mathbf{v}}(f_0), in_{\mathbf{v}}(f_1), \ldots, in_{\mathbf{v}}(f_{n-1})) = \mathbf{0}$.

Definition (Pretropism)

A **pretropism** $\mathbf{v} \in \mathbb{Z}^n$ is a vector, common to all Newton polytopes of the polynomial system. A pretropism leads to an initial form system.

Definition (Tropism)

A **tropism** is a pretropism, which is the leading exponent vector in a Puiseux series expansion of a curve, expanded about $t \approx 0$.

The Cayley Embedding & Polytope For Square Systems

We obtain pretropisms for polynomial systems using the *Cayley polytope*.

Cayley Embedding

$$C_E = (A_0 \times \{\mathbf{0}\}) \cup (A_1 \times \{\mathbf{e}_1\}) \cup \cdots \cup (A_{n-1} \times \{\mathbf{e}_{n-1}\})$$

where $\mathbf{e}_k$ is the k-th $(n-1)$-dimensional standard unit vector.

Cayley Polytope

$$C_\Delta = ConvexHull(C_E)$$

Remark

We use the Cayley polytope as a way to combine all individual Newton polytopes into one Cayley polytope and obtain their **common** facet normals.

We use **cddlib** of *K. Fukuda* to find facet normals of the Cayley polytope. Alternative: **gfan**, developed by *A.N. Jensen*, finds cones of pretropisms.

Tropisms and d-Dimensional Surfaces

For d-dimensional solution sets we have cones of tropisms.

Definition (Cone of Tropisms)

A **cone of tropisms** is a polyhedral cone, spanned by tropisms.

- $v_0,\ v_1, \ldots,\ v_{d-1}$ span a d-dimensional cone of tropisms
- dimension of the cone is the dimension of the solution set

Let $v_0 = (v_{(0,1)}, v_{(0,2)}, \ldots, v_{(0,n-1)})$, $v_1 = (v_{(1,0)}, v_{(1,1)}, \ldots, v_{(1,n-1)})$, $\ldots$, $v_{d-1} = (v_{(d-1,0)}, v_{(d-1,1)}, \ldots, v_{(d-1,n-1)})$ be d tropisms. Let $r_0, r_1, \ldots, r_{n-1}$ be the solutions of the initial form system $in_{\mathbf{v}_0}(in_{\mathbf{v}_1}(\cdots in_{\mathbf{v}_{d-1}}(F)\cdots))(\mathbf{x}) = \mathbf{0}$.

d tropisms generate a Puiseux series expansion of a d-dimensional surface

$$
\begin{aligned}
x_0 &= t_0^{v_{(0,0)}} t_1^{v_{(1,0)}} \cdots t_{d-1}^{v_{(d-1,0)}} (r_0 + c_{(0,0)} t_0^{w_{(0,0)}} + c_{(1,0)} t_1^{w_{(1,0)}} + \ldots) \\
x_1 &= t_0^{v_{(0,1)})} t_1^{v_{(1,1)}} \cdots t_{d-1}^{v_{(d-1,1)}} (r_1 + c_{(0,1)} t_0^{w_{(0,1)}} + c_{(1,1)} t_1^{w_{(1,1)}} + \ldots) \\
x_2 &= t_0^{v_{(0,2)}} t_1^{v_{(1,2)}} \cdots t_{d-1}^{v_{(d-1,2)}} (r_2 + c_{(0,2)} t_0^{w_{(0,2)}} + c_{(1,2)} t_1^{w_{(1,2)}} + \ldots) \\
&\ \vdots \\
x_{n-1} &= t_0^{v_{(0,n-1)}} t_1^{v_{(1,n-1)}} \cdots t_{d-1}^{v_{(d-1,n-1)}} (r_{n-1} + c_{(0,n-1)} t_0^{w_{(0,n-1)}} + c_{(1,n-1)} t_1^{w_{(1,n-1)}} + \ldots)
\end{aligned}
$$

Unimodular Coordinate Transformation

Definition (Unimodular Coordinate Transformation)

Let $M \in \mathbb{Z}^{n \times n}$ be a matrix with $det(M) = \pm 1$. Then, the **unimodular coordinate transformation** is a power transformation of the form $\mathbf{x} = \mathbf{z}^M$.

matrix M

- contains the d dimensional cone tropisms in their first d rows
- used to transform
  - initial form systems i.e. $in_{\mathbf{v}}(F)(\mathbf{x} = \mathbf{z}^M)) \rightarrow$ isolated solutions at infinity
  - polynomial systems $\rightarrow$ second term in the Puiseux series
- $\mathbf{x} = \mathbf{z}^M$ puts solution sets in a specific format

Our method to obtain matrix M uses the computation of:

- Smith Normal Form (for series with integer exponents)
- Hermite Normal Form (for series with fractional exponents)

Related result: E. Hubert and G. Labahn. *Rational invariants of scalings from Hermite normal forms*. In Proceedings of ISSAC 2012, pages 219226. ACM, 2012.

General Algebraic Sets

Assumptions on the solution sets we can find

Proposition *If $F(\mathbf{x}) = \mathbf{0}$ is in Noether position and defines a d-dimensional solution set in $\mathbb{C}^n$, intersecting the first d coordinate planes in regular isolated points, then there are d linearly independent tropisms $\mathbf{v}_0, \mathbf{v}_1, \ldots \mathbf{v}_{d-1} \in \mathbb{Q}^n$ so that the initial form system $in_{\mathbf{v}_0}(in_{\mathbf{v}_1}(\cdots in_{\mathbf{v}_{d-1}}(F)\cdots))(\mathbf{x} = \mathbf{z}^M) = \mathbf{0}$ has a solution* $\mathbf{c} \in (\mathbb{C} \setminus \{0\})^{n-d}$. This solution and the tropisms are the leading coefficients and powers of a generalized Puiseux series expansion for the algebraic set:

$$\begin{aligned}
x_0 &= t_0^{v_{0,0}} \\
x_1 &= t_0^{v_{0,1}} t_1^{v_{1,1}} \\
&\vdots \\
x_{d-1} &= t_0^{v_{0,d-1}} t_1^{v_{1,d-1}} \cdots t_{d-1}^{v_{d-1,d-1}} \\
x_d &= c_0 t_0^{v_{0,d}} t_1^{v_{1,d}} \cdots t_{d-1}^{v_{d-1,d}} + \cdots \\
x_{d+1} &= c_1 t_0^{v_{0,d+1}} t_1^{v_{1,d+1}} \cdots t_{d-1}^{v_{d-1,d+1}} + \cdots \\
&\vdots \\
x_n &= c_{n-d-1} t_0^{v_{0,n-1}} t_1^{v_{1,n-1}} \cdots t_{d-1}^{v_{d-1,n-1}} + \cdots
\end{aligned}$$

Cyclic n-roots Problem

$$C_n(\mathbf{x}) = \begin{cases} x_0 + x_1 + \cdots + x_{n-1} = 0 \\ x_0x_1 + x_1x_2 + \cdots + x_{n-2}x_{n-1} + x_{n-1}x_0 = 0 \\ i = 3, 4, \ldots, n-1 : \sum_{j=0}^{n-1} \prod_{k=j}^{j+i-1} x_{k \mod n} = 0 \\ x_0x_1x_2 \cdots x_{n-1} - 1 = 0. \end{cases}$$

- benchmark problem in the field of computer algebra (pop. by J. Davenport)
- extremely hard to solve for $n \geq 8$
- square systems
  - we expect isolated solutions
  - we find positive dimensional solution sets

Lemma (Backelin)

If m^2 divides n, then the dimension of the cyclic n-roots polynomial system is at least $m-1$.

J. Backelin: *Square multiples n give infinitely many cyclic n-roots.*
Reports, Matematiska Institutionen, Stockholms Universitet, 1989.

J. Davenport. *Looking at a set of equations.*
Technical Report 87-06, Bath Computer Science, 1987.

Cyclic 8-Roots System

an illustration:

for pretropism $\mathbf{v} = (1, -1, 0, 1, 0, 0, -1, 0)$, the initial form system is

$$in_{\mathbf{v}}(C_8)(\mathbf{x}) = \begin{cases} x_1 + x_6 = 0 \\ x_1x_2 + x_5x_6 + x_6x_7 = 0 \\ x_4x_5x_6 + x_5x_6x_7 = 0 \\ x_0x_1x_6x_7 + x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_6x_7 + x_0x_1x_5x_6x_7 = 0 \\ x_0x_1x_2x_5x_6x_7 + x_0x_1x_4x_5x_6x_7 + x_1x_2x_3x_4x_5x_6 = 0 \\ x_0x_1x_2x_4x_5x_6x_7 + x_1x_2x_3x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_3x_4x_5x_6x_7 - 1 = 0 \end{cases}$$

$$M = \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and the corresponding unimodular coordinate transformation $\mathbf{x} = \mathbf{z}^M$:

$$x_0 = z_0, x_1 = z_1/z_0, x_2 = z_2, x_3 = z_0z_3, x_4 = z_4, x_5 = z_5, x_6 = z_6/z_0, x_7 = z_7$$

Cyclic 8-Roots System

$$in_{\mathbf{v}}(C_8)(\mathbf{z}) = \begin{cases} z_1 + z_6 = 0 \\ z_1z_2 + z_5z_6 + z_6z_7 = 0 \\ z_4z_5z_6 + z_5z_6z_7 = 0 \\ z_4z_5z_6z_7 + z_1z_6z_7 = 0 \\ z_1z_2z_6z_7 + z_1z_5z_6z_7 = 0 \\ z_1z_2z_3z_4z_5z_6 + z_1z_2z_5z_6z_7 + z_1z_4z_5z_6z_7 = 0 \\ z_1z_2z_3z_4z_5z_6z_7 + z_1z_2z_4z_5z_6z_7 = 0 \\ z_1z_2z_3z_4z_5z_6z_7 - 1 = 0 \end{cases}$$

Solving $in_{\mathbf{v}}(C_8)(\mathbf{z})$, we obtain 8 solutions (all in the same orbit). We select

$$z_0 = t, z_1 = -i, z_2 = \frac{-1}{2} - \frac{i}{2}, z_3 = -1, z_4 = 1 + i,$$
$$z_5 = \frac{1}{2} + \frac{i}{2}, z_6 = i, z_7 = -1 - i, i = \sqrt{-1}.$$

- these are the leading coefficients in the Puiseux series of the space curve
- next step is to find the second term

Cyclic 8-Roots System

Proposition *If the initial root does not satisfy the entire transformed polynomial system, then there must be at least one nonzero constant exponent a_i, forming monomial $c_i t^{a_i}$.*

<u>illustration</u>

$$r_1 = -i, r_2 = \frac{-1}{2} - \frac{i}{2}, r_3 = -1, r_4 = 1 + i, r_5 = \frac{1}{2} + \frac{i}{2}, r_6 = i, r_7 = -1 - i$$

Substituting the form $z_i = r_i + k_i t^w$, $i = 1 \ \ldots \ n-1$, into the transformed system $C_8(\mathbf{z})$, yields

$$\begin{array}{l} t^w(...) \ + \ ... \\ t^2(...) \ + \ ... \\ t^w(...) \ + \ ... \\ 4t + t^w(...) \ + \ ... \\ t^w(...) \ + \ ... \\ t^2(...) \ + \ ... \\ t^w(...) \ + \ ... \\ t^w(...) \ + \ ... \end{array}$$

in this case $c_i t^{a_i} = 4t^1$

Cyclic 8-Roots System

Taking solution at infinity, we build a series of the form:

$$
\begin{aligned}
z_0 &= t\\
z_1 &= -I + k_1 t\\
z_2 &= \frac{-1}{2} - \frac{I}{2} + k_2 t\\
z_3 &= -1 + k_3 t\\
z_4 &= 1 + I + k_4 t\\
z_5 &= \frac{1}{2} + \frac{I}{2} + k_5 t\\
z_6 &= I + k_6 t\\
z_7 &= (-1 - I) + k_7 t
\end{aligned}
$$

Plugging series form into transformed system, collecting all coefficients of t^1 and solving, yields

$$
\begin{aligned}
k_1 &= -1 - I\\
k_2 &= \frac{1}{2}\\
k_3 &= 0\\
k_4 &= -1\\
k_5 &= \frac{-1}{2}\\
k_6 &= 1 + I\\
k_7 &= 1
\end{aligned}
$$

The second term in the series, still in the transformed coordinates:

$$
\begin{aligned}
z_0 &= t\\
z_1 &= -I + (-1 - I)t\\
z_2 &= \frac{-1}{2} - \frac{I}{2} + \frac{1}{2}t\\
z_3 &= -1\\
z_4 &= 1 + I - t\\
z_5 &= \frac{1}{2} + \frac{I}{2} - \frac{1}{2}t\\
z_6 &= I + (1 + I)t\\
z_7 &= (-1 - I) + t
\end{aligned}
$$

Cyclic 4,8,12-roots problem

Often, first term in the Puiseux satisfies the entire system:

cyclic 4-roots:
tropism: (1,-1,1,-1)
$x_0 = t$, $x_1 = t^{-1}$, $x_2 = -t$, $x_3 = -t^{-1}$

cyclic 8-roots:
tropism: (1,-1,1,-1,1,-1,1,-1)
$x_0 = t$, $x_1 = t^{-1}$, $x_2 = it$, $x_3 = it^{-1}$, $x_4 = -t$, $x_5 = -t^{-1}$,
$x_6 = -it$, $x_7 = -it^{-1}$

cyclic 12-roots:
tropism: (1,-1,1,-1, 1,-1,1,-1,1,-1,1,-1)
$x_0 = t$, $x_1 = t^{-1}$, $x_2 = (\frac{1+\sqrt{3}i}{2})t$, $x_3 = (\frac{1+\sqrt{3}i}{2})t^{-1}$,
$x_4 = (\frac{-1+\sqrt{3}i}{2})t$, $x_5 = (\frac{-1+\sqrt{3}i}{2})t^{-1}$, $x_6 = -t$, $x_7 = -t^{-1}$,
$x_8 = (\frac{-1-\sqrt{3}i}{2})t$, $x_9 = (\frac{-1-\sqrt{3}i}{2})t^{-1}$, $x_{10} = (\frac{1-\sqrt{3}i}{2})t$, $x_{11} = (\frac{1-\sqrt{3}i}{2})t^{-1}$

Observing structure among

- tropism
- coefficients
  - numerical solver PHCpack was used
  - we recognize the coefficients as $\frac{n}{2}$-*roots of unity*

Proposition

For $n = 4\ell$, there is a one-dimensional set of cyclic n-roots, represented exactly as

$$\begin{aligned} x_{2k} &= u_k t \\ x_{2k+1} &= u_k t^{-1} \end{aligned}$$

for $k = 0, \ldots, \frac{n}{2} - 1$ and $u_k = e^{\frac{i2\pi k}{\frac{n}{2}}} = e^{\frac{i4\pi k}{n}}$.

taking random linear combination of the solutions

$$\alpha_0 t + \alpha_1 t^{-1} + \alpha_2 t + \alpha_3 t^{-1} + \cdots + \alpha_{n-2} t + \alpha_{n-1} t^{-1} = 0, \quad \alpha_j \in \mathbb{C}$$

and simplifying

$$\beta_0 t^2 + \beta_1 = 0, \quad \beta_j \in \mathbb{C}$$

we see that all space curves are **quadratic**.

Cyclic 9-Roots Polynomial System

for the cyclic 9-roots system, there is a cone of pretropisms, generated by
$v_0 = (1, 1, -2, 1, 1, -2, 1, 1, -2)$
$v_1 = (0, 1, -1, 0, 1, -1, 0, 1, -1)$.

$$In_{v_1}(In_{v_0}(C_9))(\mathbf{x}) = \begin{cases} x_2 + x_5 + x_8 = 0 \\ x_0x_8 + x_2x_3 + x_5x_6 = 0 \\ x_0x_1x_2 + x_0x_1x_8 + x_0x_7x_8 + x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 \\ +x_4x_5x_6 + x_5x_6x_7 + x_6x_7x_8 = 0 \\ x_0x_1x_2x_8 + x_2x_3x_4x_5 + x_5x_6x_7x_8 = 0 \\ x_0x_1x_2x_3x_8 + x_0x_5x_6x_7x_8 + x_2x_3x_4x_5x_6 = 0 \\ x_0x_1x_2x_3x_4x_5 + x_0x_1x_2x_3x_4x_8 + x_0x_1x_2x_3x_7x_8 \\ +x_0x_1x_2x_6x_7x_8 + x_0x_1x_5x_6x_7x_8 + x_0x_4x_5x_6x_7x_8 + x_1x_2x_3x_4x_5x_6 \\ +x_2x_3x_4x_5x_6x_7 + x_3x_4x_5x_6x_7x_8 = 0 \\ x_0x_1x_2x_3x_4x_5x_8 + x_0x_1x_2x_5x_6x_7x_8 + x_2x_3x_4x_5x_6x_7x_8 = 0 \\ x_0x_1x_2x_3x_4x_5x_6x_8 + x_0x_1x_2x_3x_5x_6x_7x_8 + x_0x_2x_3x_4x_5x_6x_7x_8 = 0 \\ x_0x_1x_2x_3x_4x_5x_6x_7x_8 - 1 = 0 \end{cases}$$

For one of the first solutions of the cyclic 9-roots polynomial system, we refer to J. C. Faugère, *A new efficient algorithm for computing Gröbner bases (F_4).* Journal of Pure and Applied Algebra, Vol. 139, Number 1-3, Pages 61-88, Year 1999. Proceedings of MEGA'98, 22–27 June 1998, Saint-Malo, France.

Cyclic 9-Roots Polynomial System Cont.

$v_0 =(1, 1, -2, 1, 1, -2, 1, 1, -2)$
$v_1 =(0, 1, -1, 0, 1, -1, 0, 1, -1)$
The unimodular coordinate transformation $x = z^M$ acts on the exponents. The new coordinates are given by

$$M = \begin{bmatrix} 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & -2 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} x_0 &= z_0 \\ x_1 &= z_0 z_1 \\ x_2 &= z_0^{-2} z_1^{-1} z_2 \\ x_3 &= z_0 z_3 \\ x_4 &= z_0 z_1 z_4 \\ x_5 &= z_0^{-2} z_1^{-1} z_5 \\ x_6 &= z_0 z_6 \\ x_7 &= z_0 z_1 z_7 \\ x_8 &= z_0^{-2} z_1^{-1} z_8 \end{aligned}$$

We use the coordinate change to transform the initial form system and the original cyclic 9-roots system.

Cyclic 9-Roots Polynomial System Cont.

The transformed initial form system $in_{v_1}(in_{v_0}(C_9))(\mathbf{z})$ is given by

$$
\begin{cases}
z_2 + z_5 + z_8 = 0 \\
z_2 z_3 + z_5 z_6 + z_8 = 0 \\
z_2 z_3 z_4 + z_3 z_4 z_5 + z_4 z_5 z_6 + z_5 z_6 z_7 + z_6 z_7 z_8 + z_2 z_3 + z_7 z_8 + z_2 + z_8 = 0 \\
z_2 z_3 z_4 z_5 + z_5 z_6 z_7 z_8 + z_2 z_8 = 0 \\
z_2 z_3 z_4 z_5 z_6 + z_5 z_6 z_7 z_8 + z_2 z_3 z_8 = 0 \\
z_2 z_3 z_4 z_5 z_6 z_7 + z_3 z_4 z_5 z_6 z_7 z_8 + z_2 z_3 z_4 z_5 z_6 + z_4 z_5 z_6 z_7 z_8 + z_2 z_3 z_4 z_5 + z_2 z_3 z_4 z_8 \\
+ z_2 z_3 z_7 z_8 + z_2 z_6 z_7 z_8 + z_5 z_6 z_7 z_8 = 0 \\
z_3 z_4 z_6 z_7 + z_3 z_4 + z_6 z_7 = 0 \\
z_4 z_7 + z_4 + z_7 = 0 \\
z_2 z_3 z_4 z_5 z_6 z_7 z_8 - 1 = 0
\end{cases}
$$

Its solution is
$z_2 = -\frac{1}{2} - \frac{\sqrt{3}i}{2},\ z_3 = -\frac{1}{2} + \frac{\sqrt{3}i}{2},\ z_4 = -\frac{1}{2} + \frac{\sqrt{3}i}{2},\ z_5 = 1,\ z_6 = -\frac{1}{2} - \frac{\sqrt{3}i}{2},$
$z_7 = -\frac{1}{2} - \frac{\sqrt{3}i}{2},\ z_8 = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$, where $i = \sqrt{-1}$.

While we used a numerical solver PHCpack, we recognized the solution as the 3^{rd} roots of unity.

Cyclic 9-Roots Polynomial System Cont.

The following assignment satisfies cyclic 9-roots polynomial system **entirely**.

$$
\begin{aligned}
z_0 &= t_0\\
z_1 &= t_1\\
z_2 &= -\frac{1}{2} - \frac{\sqrt{3}i}{2}\\
z_3 &= -\frac{1}{2} + \frac{\sqrt{3}i}{2}\\
z_4 &= -\frac{1}{2} + \frac{\sqrt{3}i}{2}\\
z_5 &= 1\\
z_6 &= -\frac{1}{2} - \frac{\sqrt{3}i}{2}\\
z_7 &= -\frac{1}{2} - \frac{\sqrt{3}i}{2}\\
z_8 &= -\frac{1}{2} + \frac{\sqrt{3}i}{2}
\end{aligned}
$$

$$
\begin{aligned}
x_0 &= z_0\\
x_1 &= z_0 z_1\\
x_2 &= z_0^{-2} z_1^{-1} z_2\\
x_3 &= z_0 z_3\\
x_4 &= z_0 z_1 z_4\\
x_5 &= z_0^{-2} z_1^{-1} z_5\\
x_6 &= z_0 z_6\\
x_7 &= z_0 z_1 z_7\\
x_8 &= z_0^{-2} z_1^{-1} z_8
\end{aligned}
$$

$$
\begin{aligned}
x_0 &= t_0\\
x_1 &= t_0 t_1\\
x_2 &= t_0^{-2} t_1^{-1}\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)\\
x_3 &= t_0\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)\\
x_4 &= t_0 t_1\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)\\
x_5 &= t_0^{-2} t_1^{-1}\\
x_6 &= t_0\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)\\
x_7 &= t_0 t_1\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)\\
x_8 &= t_0^{-2} t_1^{-1}\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)
\end{aligned}
$$

Cyclic 9-Roots Polynomial System Cont.

Letting $u = e^{\frac{2\pi i}{3}}$ and $y_0 = t_0,\ y_1 = t_0 t_1,\ y_2 = t_0^{-2} t_1^{-1} u^2$
we can rewrite the exact solution as

$$\begin{array}{lll} x_0 = t_0 & x_3 = t_0 u & x_6 = t_0 u^2 \\ x_1 = t_0 t_1 & x_4 = t_0 t_2 u & x_7 = t_0 t_2 u^2 \\ x_2 = t_0^{-2} t_1^{-1} u^2 & x_5 = t_0^{-2} t_1^{-1} & x_8 = t_0^{-2} t_1^{-1} u \end{array}$$

$$\begin{array}{lll} x_0 = y_0 & x_3 = y_0 u & x_6 = y_0 u^2 \\ x_1 = y_1 & x_4 = y_1 u & x_7 = y_1 u^2 \\ x_2 = y_2 & x_5 = y_2 u & x_8 = y_2 u^2 \end{array}$$

and put it in the same format as in the proof of Backelin's Lemma, given in J. C. Faugère, *Finding all the solutions of Cyclic 9 using Gröbner basis techniques.* In Computer Mathematics: Proceedings of the Fifth Asian Symposium (ASCM), pages 1-12. World Scientific, 2001.

degree of the solution component

$$\begin{array}{l} \alpha_1 t_0 + \alpha_2 t_0 t_1 + \alpha_3 t_0^{-2} t_1^{-1} = 0 \\ \alpha_4 t_0 + \alpha_5 t_0 t_1 + \alpha_6 t_0^{-2} t_1^{-1} = 0 \end{array} \quad \alpha_i \in \mathbb{C}$$

Simplifying, the system becomes

$$\begin{array}{r} t_0^{-2} t_1^{-1} - \beta_1 = 0 \\ t_1 - \beta_2 = 0 \end{array}$$

As the simplified system has 3 solutions, the cyclic 9 solution component is a **cubic** surface. With the cyclic permutation, we obtain an orbit of 6 cubic surfaces, which satisfy the cyclic 9-roots system.

Cyclic 16-Roots Polynomial System

Extending the pattern we observed among tropisms of the cyclic 9-roots,
$v_0 = (1, 1, -2, 1, 1, -2, 1, 1, -2)$
$v_1 = (0, 1, -1, 0, 1, -1, 0, 1, -1)$
we can get the correct cone of tropisms for the cyclic 16-roots.
$v_0 = (1, 1, 1, -3, 1, 1, 1, -3, 1, 1, 1, -3, 1, 1, 1, -3)$
$v_1 = (0, 1, 1, -2, 0, 1, 1, -2, 0, 1, 1, -2, 0, 1, 1, -2)$
$v_2 = (0, 0, 1, -1, 0, 0, 1, -1, 0, 0, 1, -1, 0, 0, 1, -1)$
Extending the solutions at infinity pattern,
cyclic 9-roots: $u = e^{\frac{2\pi i}{3}}$ $\rightarrow$ cyclic 16-roots: $u = e^{\frac{2\pi i}{4}}$
The 3-dimensional solution component of the cyclic 16-roots is given by:

$$
\begin{array}{llll}
x_0 = t_0 & x_4 = ut_0 & x_8 = u^2 t_0 & x_{12} = u^3 t_0 \\
x_1 = t_0 t_1 & x_5 = ut_0 t_1 & x_9 = u^2 t_0 t_1 & x_{13} = u^3 t_0 t_1 \\
x_2 = t_0 t_1 t_2 & x_6 = ut_0 t_1 t_2 & x_{10} = u^2 t_0 t_1 t_2 & x_{14} = u^3 t_0 t_1 t_2 \\
x_3 = t_0^{-3} t_1^{-2} t_2^{-1} & x_7 = ut_0^{-3} t_1^{-2} t_2^{-1} & x_{11} = u^2 t_0^{-3} t_1^{-2} t_2^{-1} & x_{15} = u^3 t_0^{-3} t_1^{-2} t_2^{-1}
\end{array}
$$

This 3-dimensional cyclic 16-root solution component is a **quartic** surface.
Using cyclic permutation, we obtain $2 * 4 = 8$ components of degree 4.

Cyclic n-Roots Polynomial System: $n = m^2$ case

We now generalize the previous results for the cyclic n-roots systems.

Proposition *For $n = m^2$, there is a $(m-1)$-dimensional set of cyclic n-roots, represented exactly as*

$$\begin{array}{rcl} x_{km+0} & = & u_k t_0 \\ x_{km+1} & = & u_k t_0 t_1 \\ x_{km+2} & = & u_k t_0 t_1 t_2 \\ & \vdots & \\ x_{km+m-2} & = & u_k t_0 t_1 t_2 \cdots t_{m-2} \\ x_{km+m-1} & = & u_k t_0^{-m+1} t_1^{-m+2} \cdots t_{m-3}^{-2} t_{m-2}^{-1} \end{array}$$

for $k = 0, 1, 2, \ldots, m-1$ and $u_k = e^{i2k\pi/m}$.

Proposition *The $(m-1)$-dimensional solutions set has degree equal to m.*

Applying cyclic permutation, we can find $2m$ components of degree m.

Tropical Lemma of Backelin

Lemma (Backelin)

If m^2 divides n, then the dimension of the cyclic n-roots polynomial system is at least $m-1$.

Lemma (Tropical Version of Backelin's Lemma)

For $n = m^2\ell$, where $\ell \in \mathbb{N} \setminus \{0\}$ and ℓ is no multiple of k^2, for $k \geq 2$, there is an $(m-1)$-dimensional set of cyclic n-roots, represented exactly as

$$\begin{array}{rcl} x_{km+0} & = & u^k t_0 \\ x_{km+1} & = & u^k t_0 t_1 \\ x_{km+2} & = & u^k t_0 t_1 t_2 \\ & \vdots & \\ x_{km+m-2} & = & u^k t_0 t_1 t_2 \cdots t_{m-2} \\ x_{km+m-1} & = & \gamma u^k t_0^{-m+1} t_1^{-m+2} \cdots t_{m-3}^{-2} t_{m-2}^{-1} \end{array} \qquad (1)$$

for $k = 0, 1, 2, \ldots, m-1$, free parameters $t_0, t_1, \ldots, t_{m-2}$, constants $u = e^{\frac{i2\pi}{m\ell}}$, $\gamma = e^{\frac{i\pi\beta}{m\ell}}$, with $\beta = (\alpha \mod 2)$, and $\alpha = m(m\ell - 1)$.