Tropisms, Surfaces and the Puiseux Series

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Introduction

Our main objective: development of a polyhedral method to solve systems of polynomials

- solving for d-dimensional solution sets, $d \geq 2$

We focus initially on

- binomial systems, i.e. toric ideals
- unimodular coordinate transformations to work with points at infinity
    - Smith normal form
    - Hermite normal form

We then extend these ideas to

- general polynomial systems

We use

- cones of tropisms
- Puiseux series
- to connect tropisms with d-dimensional solution sets

In the process, we emphasize the *exploitation of symmetry*

Fundamental Theorem

Theorem (Fundamental Theorem of Tropical Algebraic Geometry)

$\omega \in Trop(I) \cap \mathbb{Q}^n \iff \exists p \in V(I) : -val(p) = \omega \in \mathbb{Q}^n$.

Anders Nedergaard Jensen, Hannah Markwig, Thomas Markwig: *An Algorithm for Lifting Points in a Tropical Variety*. Collect. Math. vol. 59, no. 2, pages 129–165, 2008.

Rephrasing the Theorem:

Rational vector in the tropical variety corresponds to the leading powers of a Puiseux series, converging to a point in the algebraic variety.

Our understanding of the Fundamental Theorem of Tropical Algebraic Geometry

- comes from polyhedral homotopies
- we see it as a generalization of *Bernshtein's Theorem B*
- we use *Bernshtein's Theorem A &B* to solve polynomial systems with polyhedral methods

Our approach is inspired by the Fundamental Theorem of Tropical Algebraic Geometry

Cyclic n-roots Polynomial Systems

The cyclic n-roots polynomial systems are benchmark problems for polynomial system solvers.

$$F(\mathbf{x}) = C_n(\mathbf{x}) = \begin{cases} x_0 + x_1 + \cdots + x_{n-1} = 0 \\ x_0x_1 + x_1x_2 + \cdots + x_{n-2}x_{n-1} + x_{n-1}x_0 = 0 \\ i = 3, 4, \ldots, n-1 : \displaystyle\sum_{j=0}^{n-1} \prod_{k=j}^{j+i-1} x_{k \mod n} = 0 \\ x_0x_1x_2 \cdots x_{n-1} - 1 = 0. \end{cases}$$

Cyclic n-roots polynomial systems:

- square systems: we expect isolated solutions
- we find positive dimensional solution sets

Lemma (Backelin)

If m^2 divides n, then the dimension of the cyclic n-roots polynomial system is at least $m - 1$.

J. Backelin: *Square multiples n give infinitely many cyclic n-roots*. Reports, Matematiska Institutionen, Stockholms Universitet, 1989.

Algorithm Outline for Solving of Binomial Systems

We represent a binomial system in the following way:

$$\mathbf{x}^A - \mathbf{c} = 0 \qquad \text{for i } = 0, 1, \ldots, k-1$$

where $A \in \mathbb{Z}^{k \times n}$, $\mathbf{c}$=$(c_0, c_1, \ldots, c_{k-1})^T$, $c_i \in \mathbb{C} \setminus 0$.

If the rank(A) = k, then k is the codimension of the solution set. Otherwise, $\mathbf{x}^A - \mathbf{c} = 0$ has no $(n-k)$-dimensional solution set.

For an input $(A, \ \mathbf{c})$, with rank(A) = k, we describe the solution set of the binomial system by

- a square matrix M
- computed values for the last (n-k) variables

Algorithm Outline for Solving of Binomial Systems

INPUT: $(A,\ \mathbf{c})$

1. Compute the null space B of A, $d = n - k$.
2. Compute the Smith Normal Form (U, S, V) of B.
3. Depending on U and S, do one of the following:
   3.1. If $U = I$, then $M = V^{-1}$
   3.2. If $U \neq$ I and S has 1's on the diagonal, then extend U^{-1} with an identity matrix to obtain an $n \times n$ matrix E that has U^{-1} in its first d rows. Then $M = EV^{-1}$.
   3.3. In all other cases:
      3.3.1. Compute Hermite Normal Form of B, $UB = H$, $det(U) = \pm 1$, $U \in \mathbb{Z}^{n \times n}$. We assume that B has full rank and that the columns of B have been permuted, s.t. H has only non-zero elements on its diagonal.
      3.3.2. Let D be a diagonal matrix of the same dimension as U, which takes its elements from the corresponding diagonal elements of H. Then,

$$M = \begin{bmatrix} D^{-1} & B \\ \mathbf{0} & I \end{bmatrix}$$

4. After the coordinate transformation $\mathbf{x} = \mathbf{z}^M$, solve the resulting binomial system in k equations and k unknowns. Return M and the solutions of the transformed binomial system.

Consider the binomial system:

$$\begin{cases} x_0^4 x_1^5 x_2^3 x_3^8 x_4^7 - 1 = 0 \\ x_0^{11} x_1^{10} x_2^9 x_3^4 x_4^8 - 1 = 0 \end{cases}$$

Writing the exponents in form of a matrix

$$A = \begin{bmatrix} 4 & 5 & 3 & 8 & 7 \\ 11 & 10 & 9 & 4 & 8 \end{bmatrix}$$

We are looking for the null space of A

Three linearly independent vectors satisfy $A\mathbf{v} = 0$

$$B = \begin{bmatrix} 1 & 1 & -1 & 1 & -2 \\ 2 & 1 & -4 & -1 & 1 \\ 3 & -2 & -1 & 1 & -1 \end{bmatrix} \quad AB^T = 0$$

Solving Binomial Systems

We want to generate an unimodular matrix M, whose first rows consist of the vectors of matrix B.

$$B = \begin{bmatrix} 1 & 1 & -1 & 1 & -2 \\ 2 & 1 & -4 & -1 & 1 \\ 3 & -2 & -1 & 1 & -1 \end{bmatrix} \quad AB^T = 0$$

$$M = \begin{bmatrix} 1 & 1 & -1 & 1 & -2 \\ 2 & 1 & -4 & -1 & 1 \\ 3 & -2 & -1 & 1 & -1 \\ n_{3,0} & n_{3,1} & n_{3,2} & n_{3,3} & n_{3,4} \\ n_{4,0} & n_{4,1} & n_{4,2} & n_{4,3} & n_{4,4} \end{bmatrix} \quad n_{i,j} \in \mathbb{N}$$

$$det(M) \pm 1$$

We use the matrix M as power transformation, to change the coordinates of the binomial system via:

$$\mathbf{x} = \mathbf{z}^M$$

The Smith Normal Form: $S = UBV$ or $B = U^{-1}SV^{-1}$

$$B = \begin{bmatrix} 1 & 1 & -1 & 1 & -2 \\ 2 & 1 & -4 & -1 & 1 \\ 3 & -2 & -1 & 1 & -1 \end{bmatrix} \quad AB^T = 0$$

Computing the Smith Normal Form of B with Sage, yields matrices

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad det(U) = \pm 1$$

$$V = \begin{bmatrix} 7 & 6 & -1 & 7 & 15 \\ 5 & 4 & -1 & 5 & 10 \\ 6 & 5 & -1 & 6 & 13 \\ -5 & -5 & 1 & -4 & -12 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad det(V) = \pm 1$$

Solving Binomial Systems

We can compute the unimodular matrix M, using the Smith Normal Form computation. Because

- S had only 1's on the diagonal
- U = I, the identity matrix

The unimodular matrix $M = V^{-1}$

$$M = V^{-1} = \begin{bmatrix} 1 & 1 & -1 & 1 & -2 \\ 2 & 1 & -4 & -1 & 1 \\ 3 & -2 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 2 & 0 & 0 \end{bmatrix} \quad det(M) = \pm 1$$

By having $v_0,\ v_1,\ v_2 \in B$ in the first three rows of M, we will eliminate the first three variables in the binomial system, after the unimodular coordinate transformation.

Solving Binomial Systems

The unimodular coordinate transformation acts on the exponents: $\mathbf{x} = \mathbf{z}^M$.

$$M = \begin{bmatrix} 1 & 1 & -1 & 1 & -2 \\ 2 & 1 & -4 & -1 & 1 \\ 3 & -2 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 2 & 0 & 0 \end{bmatrix} \quad \left\{ \begin{array}{l} x_0 = z_0 z_1^2 z_2^3 z_4^{-1} \\ x_1 = z_0 z_1 z_2^{-2} z_4^{-1} \\ x_2 = z_0^{-1} z_1^{-4} z_2^{-1} z_4^2 \\ x_3 = z_0 z_1^{-1} z_2 \\ x_4 = z_0^{-2} z_1 z_2^{-1} z_3 \end{array} \right.$$

$$\left\{ \begin{array}{l} x_0^4 x_1^5 x_2^3 x_3^8 x_4^7 - 1 = 0 \\ x_0^{11} x_1^{10} x_2^9 x_3^4 x_4^8 - 1 = 0 \end{array} \right. \qquad \left\{ \begin{array}{l} z_3^7 z_4^{-3} - 1 = 0 \\ z_3^8 z_4^{-3} - 1 = 0 \end{array} \right.$$

Solving the transformed system yields 3 isolated solutions in $\mathbb{C} \setminus 0$ for the variables z_3, z_4. Returning these solutions to original coordinates, via the transformation, we obtain the representations of the three-dimensional solution set of the original binomial system.

Solving Binomial Systems

Consider the binomial system:

$$\begin{cases} x_0 x_1^2 x_2^3 x_3 x_4^7 - 1 = 0 \\ x_0^3 x_1^5 x_2^8 x_3 x_4^2 - 1 = 0 \end{cases}$$

Writing the exponents in form of a matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 7 \\ 3 & 5 & 8 & 1 & 2 \end{bmatrix}$$

We are looking for the null space of A
Three linearly independent vectors satisfy $A\mathbf{v} = 0$

$$B = \begin{bmatrix} 6 & 6 & -7 & 10 & -1 \\ 1 & -14 & 9 & -7 & 1 \\ 2 & -3 & 1 & 1 & 0 \end{bmatrix} \quad AB^T = 0$$

Solving Binomial Systems

We want to generate an unimodular matrix M, whose first rows consist of the vectors of matrix B.

$$B = \begin{bmatrix} 6 & 6 & -7 & 10 & -1 \\ 1 & -14 & 9 & -7 & 1 \\ 2 & -3 & 1 & 1 & 0 \end{bmatrix} \quad AB^T = 0$$

$$M = \begin{bmatrix} 6 & 6 & -7 & 10 & -1 \\ 1 & -14 & 9 & -7 & 1 \\ 2 & -3 & 1 & 1 & 0 \\ n_{3,0} & n_{3,1} & n_{3,2} & n_{3,3} & n_{3,4} \\ n_{4,0} & n_{4,1} & n_{4,2} & n_{4,3} & n_{4,4} \end{bmatrix} \quad n_{i,j} \in \mathbb{N}$$

$$det(M) \pm 1$$

We use the matrix M as power transformation, to change the coordinates of the binomial system via:

$$\mathbf{x} = \mathbf{z}^M$$

Solving Binomial Systems

The Smith Normal Form: $S = UBV$ or $B = U^{-1}SV^{-1}$

$$B = \begin{bmatrix} 6 & 6 & -7 & 10 & -1 \\ 1 & -14 & 9 & -7 & 1 \\ 2 & -3 & 1 & 1 & 0 \end{bmatrix} \quad AB^T = 0$$

Computing the Smith Normal Form of B with Sage, yields matrices

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad det(U) = \pm 1$$

$$V = \begin{bmatrix} 3 & 14 & -5 & -8 & 19 \\ 4 & 25 & -9 & -13 & 31 \\ 6 & 42 & -15 & -21 & 50 \\ 0 & 6 & -2 & -2 & 5 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad det(V) = \pm 1$$

While S has 1's on the diagonal, $U \neq I \rightarrow M \neq V^{-1}$

Solving Binomial Systems

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad det(U) = \pm 1$$

Inverting matrix U and extending matrix $U^{-1} \rightarrow$ E:

$$U^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad det(E) = \pm 1$$

We can obtain the unimodular matrix M, via

$$M = EV^{-1}$$

Solving Binomial Systems

$$V = \begin{bmatrix} 3 & 14 & -5 & -8 & 19 \\ 4 & 25 & -9 & -13 & 31 \\ 6 & 42 & -15 & -21 & 50 \\ 0 & 6 & -2 & -2 & 5 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M = EV^{-1} = \begin{bmatrix} 6 & 6 & -7 & 10 & -1 \\ 1 & -14 & 9 & -7 & 1 \\ 2 & -3 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 6 & -4 & 3 & 0 \end{bmatrix}$$

By having $v_0,\ v_1,\ v_2 \in B$ in the first three rows of M, we will eliminate the first three variables in the binomial system, after the unimodular coordinate transformation.

Solving Binomial Systems

The unimodular coordinate transformation acts on the exponents: $\mathbf{x} = \mathbf{z}^M$.

$$M = \begin{bmatrix} 6 & 6 & -7 & 10 & -1 \\ 1 & -14 & 9 & -7 & 1 \\ 2 & -3 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 6 & -4 & 3 & 0 \end{bmatrix} \qquad \left\{ \begin{array}{l} x_0 = z_0^6 z_1 z_2^2 \\ x_1 = z_0^6 z_1^{-14} z_2^{-3} z_4^6 \\ x_2 = z_0^{-7} z_1^9 z_2 z_4^{-4} \\ x_3 = z_0^{10} z_1^{-7} z_2 z_4^3 \\ x_4 = z_0^{-1} z_1 z_3 \end{array} \right.$$

$$\left\{ \begin{array}{l} x_0 x_1^2 x_2^3 x_3 x_4^7 - 1 = 0 \\ x_0^3 x_1^5 x_2^8 x_3 x_4^2 - 1 = 0 \end{array} \right. \quad \rightarrow \quad \left\{ \begin{array}{l} z_3^7 z_4^3 - 1 = 0 \\ z_3^2 z_4 - 1 = 0 \end{array} \right.$$

Solving the transformed system yields one solution $z_3 = 1$, $z_4 = 1$.
Returning this solution to the original coordinates, we obtain a solution of the original binomial system.

$$\left\{ \begin{array}{l} x_0 = z_0^6 z_1 z_2^2 \\ x_1 = z_0^6 z_1^{-14} z_2^{-3} \\ x_2 = z_0^{-7} z_1^9 z_2 \\ x_3 = z_0^{10} z_1^{-7} z_2 \\ x_4 = z_0^{-1} z_1 \end{array} \right.$$

Consider the binomial system:

$$\begin{cases} x_0^5 x_1^1 x_2^2 x_3^3 x_4^2 - 1 = 0 \\ x_0^1 x_1^1 x_2^1 x_3^1 x_4^2 - 1 = 0 \\ x_0^1 x_1^1 x_2^5 x_3^1 x_4^7 - 1 = 0 \end{cases}$$

Writing the exponents in form of a matrix

$$A = \begin{bmatrix} 5 & 1 & 2 & 3 & 2 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 5 & 1 & 7 \end{bmatrix}$$

We are looking for the null space of A

Two linearly independent vectors satisfy $A\mathbf{v} = 0$

$$B = \begin{bmatrix} 2 & -9 & -10 & 1 & 8 \\ 15 & -7 & -20 & -20 & 16 \end{bmatrix} \quad AB^T = 0$$

We want to generate an unimodular matrix M, whose first rows consist of the vectors of matrix B.

$$B = \begin{bmatrix} 2 & -9 & -10 & 1 & 8 \\ 15 & -7 & -20 & -20 & 16 \end{bmatrix} \quad AB^T = 0$$

$$M = \begin{bmatrix} 2 & -9 & -10 & 1 & 8 \\ 15 & -7 & -20 & -20 & 16 \\ n_{2,0} & n_{2,1} & n_{2,2} & n_{2,3} & n_{2,4} \\ n_{3,0} & n_{3,1} & n_{3,2} & n_{3,3} & n_{3,4} \\ n_{4,0} & n_{4,1} & n_{4,2} & n_{4,3} & n_{4,4} \end{bmatrix} \quad n_{i,j} \in \mathbb{N}$$

$$det(M) \pm 1$$

We use the matrix M as a power transformation, to change the coordinates of the binomial system via:

$$\mathbf{x} = \mathbf{z}^M$$

The Smith Normal Form: $S = UBV$ or $B = U^{-1}SV^{-1}$

$$B = \begin{bmatrix} 2 & -9 & -10 & 1 & 8 \\ 15 & -7 & -20 & -20 & 16 \end{bmatrix} \quad AB^T = 0$$

Computing the Smith Normal Form of B with Sage, yields matrices

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \quad det(U) = \pm 1$$

$$V = \begin{bmatrix} 11 & -1 & -88 & -13 & 10 \\ -11 & 2 & 88 & 15 & -10 \\ 12 & -2 & -96 & -16 & 11 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad det(V) = \pm 1$$

Solving Binomial Systems

We can not compute the unimodular matrix M, using the Smith Normal Form

- S did not have only 1's on the diagonal
- U was not the identity matrix

We can use the Hermite Normal Form to rescale elements of matrix B:

$$B = \begin{bmatrix} 2 & -9 & -10 & 1 & 8 \\ 15 & -7 & -20 & -20 & 16 \end{bmatrix} \quad AB^T = 0$$

Hermite Normal Form of B:

$$H = \begin{bmatrix} 1 & 56 & 50 & -27 & -40 \\ 0 & 121 & 110 & -55 & -88 \end{bmatrix}$$

$$M = \begin{bmatrix} 2 & -9 & -10 & 1 & 8 \\ \frac{15}{121} & -\frac{7}{121} & -\frac{20}{121} & -\frac{20}{121} & \frac{16}{121} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad det(M) = \pm 1$$

$$M = \begin{bmatrix} 2 & -9 & -10 & 1 & 8 \\ \frac{15}{121} & -\frac{7}{121} & -\frac{20}{121} & -\frac{20}{121} & \frac{16}{121} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The unimodular coordinate transformation acts on the exponents: $\mathbf{x} = \mathbf{z}^M$.

$$\begin{cases} x_0 = z_0^2 z_1^{\frac{15}{121}} \\ x_1 = z_0^{-9} z_1^{-\frac{7}{121}} \\ x_2 = z_0^{-10} z_1^{-\frac{20}{121}} z_2^1 \\ x_3 = z_0^1 z_1^{-\frac{20}{121}} z_3^1 \\ x_4 = z_0^8 z_1^{\frac{16}{121}} z_4^1 \end{cases}$$

Using the new coordinates

$$\left\{ \begin{array}{l} x_0 = z_0^2 z_1^{\frac{15}{121}} \\ x_1 = z_0^{-9} z_1^{-\frac{7}{121}} \\ x_2 = z_0^{-10} z_1^{-\frac{20}{121}} z_2^1 \\ x_3 = z_0^1 z_1^{-\frac{20}{121}} z_3^1 \\ x_4 = z_0^8 z_1^{\frac{16}{121}} z_4^1 \end{array} \right. \quad (1)$$

we transform the binomial system

$$\left\{ \begin{array}{l} x_0^5 x_1^1 x_2^2 x_3^3 x_4^2 - 1 = 0 \\ x_0^1 x_1^1 x_2^1 x_3^1 x_4^2 - 1 = 0 \\ x_0^1 x_1^1 x_2^5 x_3^1 x_4^7 - 1 = 0 \end{array} \right. \rightarrow \left\{ \begin{array}{l} z_2^2 z_3^3 z_4^2 - 1 = 0 \\ z_2 z_3 z_4^2 - 1 = 0 \\ z_2^5 z_3 z_4^7 - 1 = 0 \end{array} \right.$$

Solving the transformed system yields 11 isolated solutions for the variables z_2 , z_3, z_4. Returning these solutions to original coordinates via the transformation (1), we obtain the fractional representations of the two-dimensional solution set of the original binomial system.

Polynomial System: Basic Definitions

We want to extend the method we used on binomial systems to general polynomial systems.

Polynomial System

$$F(\mathbf{x}) = \begin{cases} f_0(\mathbf{x}) = 0 \\ f_1(\mathbf{x}) = 0 \\ \vdots \\ f_{n-1}(\mathbf{x}) = 0 \end{cases} \qquad \mathbf{x} = (x_0, x_1, \ldots, x_{n-1}),\ f_i \in \mathbb{C}[\mathbf{x}]$$

A Polynomial

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad c_{\mathbf{a}} \in \mathbb{C} \setminus 0, \quad \mathbf{x}^{\mathbf{a}} = x_0^{a_0} x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}$$

The set A_i of exponents is called the support of f_i.
$P_i = \mathrm{ConvexHull}(A_i)$ is the Newton polytope of f_i.

The Cayley Embedding & Polytope For Square Systems

Cayley Embedding

$$C_E = (A_0 \times \{\mathbf{0}\}) \cup (A_1 \times \{\mathbf{e}_1\}) \cup \cdots \cup (A_{n-1} \times \{\mathbf{e}_{n-1}\})$$

where $\mathbf{e}_k$ is the k-th $(n-1)$-dimensional unit vector.

Cayley Polytope

$$C_\Delta = ConvexHull(C_E)$$

NOTE

We use the Cayley polytope as a way to combine all individual polytopes into one polytope.

We use **cddlib** of *K. Fukuda* to find facet normals of the Cayley polytope.

Tropisms and Initial Form Systems

Definition (Pretropism)

A **pretropism** is a normal vector (a facet normal) to at least an edge of each polytope.

Definition (Initial Form)

Let f_i be a polynomial with support A_i and let $\mathbf{v}$ be a pretropism. Then the **initial form** $in_{\mathbf{v}}(f_i)$ is the sum of all monomials in f_i, where the inner product $\langle \mathbf{a}, \mathbf{v} \rangle$ reaches its minimum at least twice over $\mathbf{a} \in A_i$.

Initial Form System

For a system $F(\mathbf{x}) = \mathbf{0}$, $F = (f_0, f_1, \ldots, f_{n-1})$, and pretropism $\mathbf{v}$, the **initial form system** is defined by $in_{\mathbf{v}}(F) = (in_{\mathbf{v}}(f_0), in_{\mathbf{v}}(f_1), \ldots, in_{\mathbf{v}}(f_{n-1}))$.

Solving initial form system leads to solutions at infinity.

Definition (Tropism)

A **tropism** is a pretropism, which is the leading exponent vector of a Puiseux series expansion for a curve, expanded about $t \approx 0$.

Tropisms and d-Dimensional Surfaces

Let $v_0 = (v_{(0,1)}, v_{(0,2)}, \ldots, v_{(0,n-1)})$, $v_1 = (v_{(1,0)}, v_{(1,1)}, \ldots, v_{(1,n-1)})$, $\ldots$, $v_{d-1} = (v_{(d-1,0)}, v_{(d-1,1)}, \ldots, v_{(d-1,n-1)})$ be d tropisms:

d tropisms generate a Puiseux series expansion of a d-dimensional surface

$$
\begin{aligned}
x_0 &= t_0^{v_{(0,0)}} t_1^{v_{(1,0)}} \cdots t_{d-1}^{v_{(d-1,0)}} (r_0 + c_{(0,0)} t_0^{w_{(0,0)}} + c_{(1,0)} t_1^{w_{(1,0)}} + \ldots) \\
x_1 &= t_0^{v_{(0,1)}} t_1^{v_{(1,1)}} \cdots t_{d-1}^{v_{(d-1,1)}} (r_1 + c_{(0,1)} t_0^{w_{(0,1)}} + c_{(1,1)} t_1^{w_{(1,1)}} + \ldots) \\
x_2 &= t_0^{v_{(0,2)}} t_1^{v_{(1,2)}} \cdots t_{d-1}^{v_{(d-1,2)}} (r_2 + c_{(0,2)} t_0^{w_{(0,2)}} + c_{(1,2)} t_1^{w_{(1,2)}} + \ldots) \\
&\vdots \\
x_{n-1} &= t_0^{v_{(0,n-1)}} t_1^{v_{(1,n-1)}} \cdots t_{d-1}^{v_{(d-1,n-1)}} (r_{n-1} + c_{(0,n-1)} t_0^{w_{(0,n-1)}} + c_{(1,n-1)} t_1^{w_{(1,n-1)}} + \ldots)
\end{aligned}
$$

- $v_0, v_1, \ldots, v_{d-1}$ span a **cone** of tropisms.
- dimension of the cone is d, i.e. the number of free parameters.
- r_i are the solutions of initial forms, i.e. solutions at infinity.
- $t_j \approx 0$ - our Puiseux series are valid around zero.

As an example, we will consider the cyclic 9-roots polynomial system.

We search for candidates for the cones of tropisms in the following way:

1. embed the polynomial system via the Cayley embedding
2. compute the Cayley polytope, i.e. H-rep. of the embedded system
3. remove the embedding from the Cayley polytope
4. determine which facet normals are pretropisms
5. for each pretropism, compute the initial form system
    5.1. repeat steps 1. - 5. for each initial form until there are no pretropisms, keeping track of the sequence of pretropisms, which lead to initial form systems.
6. Return each such sequence.

One such sequence is $v_0 = (1, 1, -2, 1, 1, -2, 1, 1, -2)$,
$v_1 = (0, 1, -1, 0, 1, -1, 0, 1, -1)$.

Cyclic 9-Roots Polynomial System

The cone of pretropisms for the cyclic 9-roots polynomial system was generated by vectors $v_0 = (1,1,-2,1,1,-2,1,1,-2)$ and $v_1 = (0,1,-1,0,1,-1,0,1,-1)$. Computing initial form $in_{v_0}(C_9)(\mathbf{x})$, and then $in_{v_1}(in_{v_0}(C_9))(\mathbf{x})$ yields a system:

$$
In_{v_1}(In_{v_0}(C_9))(\mathbf{x}) = \begin{cases}
x_2 + x_5 + x_8 = 0 \\
x_0x_8 + x_2x_3 + x_5x_6 = 0 \\
x_0x_1x_2 + x_0x_1x_8 + x_0x_7x_8 + x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 \\
+x_4x_5x_6 + x_5x_6x_7 + x_6x_7x_8 = 0 \\
x_0x_1x_2x_8 + x_2x_3x_4x_5 + x_5x_6x_7x_8 = 0 \\
x_0x_1x_2x_3x_8 + x_0x_5x_6x_7x_8 + x_2x_3x_4x_5x_6 = 0 \\
x_0x_1x_2x_3x_4x_5 + x_0x_1x_2x_3x_4x_8 + x_0x_1x_2x_3x_7x_8 \\
+x_0x_1x_2x_6x_7x_8 + x_0x_1x_5x_6x_7x_8 + x_0x_4x_5x_6x_7x_8 + x_1x_2x_3x_4x_5x_6 \\
+x_2x_3x_4x_5x_6x_7 + x_3x_4x_5x_6x_7x_8 = 0 \\
x_0x_1x_2x_3x_4x_5x_8 + x_0x_1x_2x_5x_6x_7x_8 + x_2x_3x_4x_5x_6x_7x_8 = 0 \\
x_0x_1x_2x_3x_4x_5x_6x_8 + x_0x_1x_2x_3x_5x_6x_7x_8 + x_0x_2x_3x_4x_5x_6x_7x_8 = 0 \\
x_0x_1x_2x_3x_4x_5x_6x_7x_8 - 1 = 0
\end{cases}
$$

For one of the first solutions of the cyclic 9-roots polynomial system, we refer to J. C. Faugère, *A new efficient algorithm for computing Gröbner bases (F_4).* Journal of Pure and Applied Algebra, Vol. 139, Number 1-3, Pages 61-88, Year 1999. Proceedings of MEGA'98, 22–27 June 1998, Saint-Malo, France.

$v_0 =$(1, 1, -2, 1, 1, -2, 1, 1, -2)
$v_1 =$(0, 1, -1, 0, 1, -1, 0, 1, -1)
The unimodular coordinate transformation $x = z^M$ acts on the exponents.
The new coordinates are given by

$$M = \begin{bmatrix} 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & -2 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} x_0 &= z_0 \\ x_1 &= z_0 z_1 \\ x_2 &= z_0^{-2} z_1^{-1} z_2 \\ x_3 &= z_0 z_3 \\ x_4 &= z_0 z_1 z_4 \\ x_5 &= z_0^{-2} z_1^{-1} z_5 \\ x_6 &= z_0 z_6 \\ x_7 &= z_0 z_1 z_7 \\ x_8 &= z_0^{-2} z_1^{-1} z_8 \end{aligned}$$

We use the coordinate change to transform the initial form system and the original cyclic 9-roots system.

Cyclic 9-Roots Polynomial System Cont.

The transformed initial form system $in_{v_1}(in_{v_0}(C_9))(\mathbf{z})$ is given by

$$\begin{cases} z_2 + z_5 + z_8 = 0 \\ z_2z_3 + z_5z_6 + z_8 = 0 \\ z_2z_3z_4 + z_3z_4z_5 + z_4z_5z_6 + z_5z_6z_7 + z_6z_7z_8 + z_2z_3 + z_7z_8 + z_2 + z_8 = 0 \\ z_2z_3z_4z_5 + z_5z_6z_7z_8 + z_2z_8 = 0 \\ z_2z_3z_4z_5z_6 + z_5z_6z_7z_8 + z_2z_3z_8 = 0 \\ z_2z_3z_4z_5z_6z_7 + z_3z_4z_5z_6z_7z_8 + z_2z_3z_4z_5z_6 + z_4z_5z_6z_7z_8 + z_2z_3z_4z_5 + z_2z_3z_4z_8 \\ +z_2z_3z_7z_8 + z_2z_6z_7z_8 + z_5z_6z_7z_8 = 0 \\ z_3z_4z_6z_7 + z_3z_4 + z_6z_7 = 0 \\ z_4z_7 + z_4 + z_7 = 0 \\ z_2z_3z_4z_5z_6z_7z_8 - 1 = 0 \end{cases}$$

Its solution is
$z_2 = -\frac{1}{2} - \frac{\sqrt{3}I}{2},\ z_3 = -\frac{1}{2} + \frac{\sqrt{3}I}{2},\ z_4 = -\frac{1}{2} + \frac{\sqrt{3}I}{2},\ z_5 = 1,\ z_6 = -\frac{1}{2} - \frac{\sqrt{3}I}{2},$
$z_7 = -\frac{1}{2} - \frac{\sqrt{3}I}{2},\ z_8 = -\frac{1}{2} + \frac{\sqrt{3}I}{2}$, where $I = \sqrt{-1}$.

While we used a numerical solver PHCpack, we recognized the solution as the 3^{rd} roots of unity.

The following assignment satisfies cyclic 9-roots polynomial system **entirely**.

$$x_0 = z_0$$
$$x_1 = z_0 z_1$$
$$x_2 = z_0^{-2} z_1^{-1} z_2$$
$$x_3 = z_0 z_3$$
$$x_4 = z_0 z_1 z_4$$
$$x_5 = z_0^{-2} z_1^{-1} z_5$$
$$x_6 = z_0 z_6$$
$$x_7 = z_0 z_1 z_7$$
$$x_8 = z_0^{-2} z_1^{-1} z_8$$

$$z_0 = t_1$$
$$z_1 = t_2$$
$$z_2 = -\frac{1}{2} - \frac{\sqrt{3}I}{2}$$
$$z_3 = -\frac{1}{2} + \frac{\sqrt{3}I}{2}$$
$$z_4 = -\frac{1}{2} + \frac{\sqrt{3}I}{2}$$
$$z_5 = 1$$
$$z_6 = -\frac{1}{2} - \frac{\sqrt{3}I}{2}$$
$$z_7 = -\frac{1}{2} - \frac{\sqrt{3}I}{2}$$
$$z_8 = -\frac{1}{2} + \frac{\sqrt{3}I}{2}$$

$$x_0 = t_1$$
$$x_1 = t_1 t_2$$
$$x_2 = t_1^{-2} t_2^{-1}\left(-\frac{1}{2} - \frac{\sqrt{3}I}{2}\right)$$
$$x_3 = t_1\left(-\frac{1}{2} + \frac{\sqrt{3}I}{2}\right)$$
$$x_4 = t_1 t_2\left(-\frac{1}{2} + \frac{\sqrt{3}I}{2}\right)$$
$$x_5 = t_1^{-2} t_2^{-1}$$
$$x_6 = t_1\left(-\frac{1}{2} - \frac{\sqrt{3}I}{2}\right)$$
$$x_7 = t_1 t_2\left(-\frac{1}{2} - \frac{\sqrt{3}I}{2}\right)$$
$$x_8 = t_1^{-2} t_2^{-1}\left(-\frac{1}{2} + \frac{\sqrt{3}I}{2}\right)$$

Cyclic 9-Roots Polynomial System Cont.

Letting $u = e^{\frac{2\pi I}{3}}$ and $y_0 = t_0, \; y_1 = t_0 t_1, \; y_2 = t_0^{-2} t_1^{-1} u^2$

$$
\begin{aligned}
x_0 &= t_1 \\
x_1 &= t_1 t_2 \\
x_2 &= t_1^{-2} t_2^{-1} \left(-\frac{1}{2} - \frac{\sqrt{3}I}{2}\right) \\
x_3 &= t_1 \left(-\frac{1}{2} + \frac{\sqrt{3}I}{2}\right) \\
x_4 &= t_1 t_2 \left(-\frac{1}{2} + \frac{\sqrt{3}I}{2}\right) \\
x_5 &= t_1^{-2} t_2^{-1} \\
x_6 &= t_1 \left(-\frac{1}{2} - \frac{\sqrt{3}I}{2}\right) \\
x_7 &= t_1 t_2 \left(-\frac{1}{2} - \frac{\sqrt{3}I}{2}\right) \\
x_8 &= t_1^{-2} t_2^{-1} \left(-\frac{1}{2} + \frac{\sqrt{3}I}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
x_0 &= t_1 \\
x_1 &= t_1 t_2 \\
x_2 &= t_1^{-2} t_2^{-1} u^2 \\
x_3 &= t_1 u \\
x_4 &= t_1 t_2 u \\
x_5 &= t_1^{-2} t_2^{-1} \\
x_6 &= t_1 u^2 \\
x_7 &= t_1 t_2 u^2 \\
x_8 &= t_1^{-2} t_2^{-1} u
\end{aligned}
$$

$$
\begin{aligned}
x_0 &= y_0 \\
x_1 &= y_1 \\
x_2 &= y_2 \\
x_3 &= y_0 u \\
x_4 &= y_1 u \\
x_5 &= y_2 u \\
x_6 &= y_0 u^2 \\
x_7 &= y_1 u^2 \\
x_8 &= y_2 u^2
\end{aligned}
$$

Cyclic 9-Roots Polynomial System Cont.

$$
\begin{aligned}
x_0 &= t_1 & x_3 &= t_1 u & x_6 &= t_1 u^2 \\
x_1 &= t_1 t_2 & x_4 &= t_1 t_2 u & x_7 &= t_1 t_2 u^2 \\
x_2 &= t_1^{-2} t_2^{-1} u^2 & x_5 &= t_1^{-2} t_2^{-1} & x_8 &= t_1^{-2} t_2^{-1} u
\end{aligned}
$$

Using this representation of the solution for points on the surface, we can compute the degree of the surface by using two random hyperplanes in the following way:

$$
\begin{aligned}
\alpha_1 t_0 + \alpha_2 t_0 t_1 + \alpha_3 t_0^{-2} t_1^{-1} &= 0 \\
\alpha_4 t_0 + \alpha_5 t_0 t_1 + \alpha_6 t_0^{-2} t_1^{-1} &= 0
\end{aligned}
\quad \alpha_i \in \mathbb{C}
$$

Simplifying, the system becomes

$$
\begin{aligned}
t_0^{-2} t_1^{-1} - \beta_1 &= 0 \\
t_1 - \beta_2 &= 0
\end{aligned}
\quad \beta_j \in \mathbb{C}
$$

As the simplified system has 3 solutions, the cyclic 9 solution component is a cubic surface.

Cyclic 9-Roots Polynomial System Cont.

Using the alternative solution format we gave earlier

$$
\begin{aligned}
x_0 &= y_0 & x_3 &= y_0 u & x_6 &= y_0 u^2 \\
x_1 &= y_1 & x_4 &= y_1 u & x_7 &= y_1 u^2 \\
x_2 &= y_2 & x_5 &= y_2 u & x_8 &= y_2 u^2
\end{aligned}
$$

we can use the cyclic permutation (forward, backward) of the third roots of unity $u = e^{\frac{2\pi i}{3}}$

$$
\begin{array}{ccc}
1 & u & u^2 \\
u & u^2 & 1 \\
u^2 & 1 & u \\
u^2 & u & 1 \\
u & 1 & u^2 \\
1 & u^2 & u
\end{array} \tag{2}
$$

and obtain an orbit of 6 cubic surfaces, satisfying the cyclic 9-roots system.

Cyclic 16-Roots Polynomial System

Extending the pattern we observed among tropisms of the cyclic 9-roots,
$v_0 = (1, 1, -2, 1, 1, -2, 1, 1, -2)$
$v_1 = (0, 1, -1, 0, 1, -1, 0, 1, -1)$
we can get the correct cone of tropisms for the cyclic 16-roots.
$v_0 = (1, 1, 1, -3, 1, 1, 1, -3, 1, 1, 1, -3, 1, 1, 1, -3)$
$v_1 = (0, 1, 1, -2, 0, 1, 1, -2, 0, 1, 1, -2, 0, 1, 1, -2)$
$v_2 = (0, 0, 1, -1, 0, 0, 1, -1, 0, 0, 1, -1, 0, 0, 1, -1)$
Extending the solutions at infinity pattern,
cyclic 9-roots: $u = e^{\frac{2\pi i}{3}}$ $\rightarrow$ cyclic 16-roots: $u = e^{\frac{2\pi i}{4}}$
The 3-dimensional solution component of the cyclic 16-roots is given by:

$x_0 = t_0$	$x_4 = ut_0$	$x_8 = u^2 t_0$	$x_{12} = u^3 t_0$
$x_1 = t_0 t_1$	$x_5 = ut_0 t_1$	$x_9 = u^2 t_0 t_1$	$x_{13} = u^3 t_0 t_1$
$x_2 = t_0 t_1 t_2$	$x_6 = ut_0 t_1 t_2$	$x_{10} = u^2 t_0 t_1 t_2$	$x_{14} = u^3 t_0 t_1 t_2$
$x_3 = t_0^{-3} t_1^{-2} t_2^{-1}$	$x_7 = ut_0^{-3} t_1^{-2} t_2^{-1}$	$x_{11} = u^2 t_0^{-3} t_1^{-2} t_2^{-1}$	$x_{15} = u^3 t_0^{-3} t_1^{-2} t_2^{-1}$

This 3-dimensional cyclic 16-root solution component is a **quartic** surface.
Using cyclic permutation, we obtain $2 * 4 = 8$ components of degree 4.

Cyclic n-Roots Polynomial System Summary

We now formalize the previous results for the cyclic n-roots systems.
Consider the cyclic n-roots polynomial systems and let $n = m^2$. Then

- there is an $(m-1)$-dimensional set of cyclic n-roots, represented exactly as

$$\begin{array}{rcl} x_{km+0} & = & u_k t_0 \\ x_{km+1} & = & u_k t_0 t_1 \\ x_{km+2} & = & u_k t_0 t_1 t_2 \\ & \vdots & \\ x_{km+m-2} & = & u_k t_0 t_1 t_2 \cdots t_{m-2} \\ x_{km+m-1} & = & u_k t_0^{-m+1} t_1^{-m+2} \cdots t_{m-3}^{-2} t_{m-2}^{-1} \end{array} \quad (3)$$

  for $k = 0, 1, 2, \ldots, m-1$ and $u_k = e^{i2k\pi/m}$.

- the $(m-1)$ dimensional solution set of (3):
  - has degree equal to m
  - there are $2m$ components of degree m

We formally address all these results in:
Computing Puiseux Series for Algebraic Surfaces
Accepted for publication in the proceedings of ISSAC 2012.

Conclusion

With the computational results, illustrated on the cyclic n-roots polynomial systems, we offer a proof of concept for a new polyhedral method to compute algebraic sets.

For more information on our polyhedral method, see

Computing Puiseux Series for Algebraic Surfaces.
`arXiv:1201.3401v2 [cs.SC]`. Accepted for publication in the proceedings of ISSAC 2012.

Polyhedral Methods for Space Curves Exploiting Symmetry.
`arXiv:1109.0241v1 [math.NA]`

Tropical Algebraic Geometry in Maple, a preprocessing algorithm for finding common factors to multivariate polynomials with approximate coefficients. Journal of Symbolic Computation 46(7):755-772, 2011.