Tropical Algebraic Geometry in Maple
a preprocessing algorithm for finding common factors to multivariate polynomials with approximate coefficients*

Danko Adrovic† Jan Verschelde‡

to Keith Geddes, on his 60^{th} *birthday*

Abstract

Finding a common factor of two multivariate polynomials with approximate coefficients is a problem in symbolic-numeric computing. Taking a tropical view of this problem leads to efficient preprocessing techniques, applying polyhedral methods on the exact exponents with numerical techniques on the approximate coefficients. With Maple we will illustrate our use of tropical algebraic geometry.

1 Introduction

Tropical algebraic geometry is a relatively new language to study skeletons of algebraic varieties. Introductions to tropical algebraic geometry are in [54] and [62, Chapter 9]. Computational aspects are addressed in [6] and [65]. One goal of this paper is to explain some new words of this language, and to show how a general purpose computer algebra system like Maple is useful to explore and illustrate tropical algebraic geometry. For software dedicated to tropical geometry, we refer to Gfan [27, 28], a SINGULAR library [29], and TrIm [63].

The roots of tropical algebraic geometry run as deep as the work of Puiseux [52] and Ostrowski [47], therefore our focus is on answering a practical question in computer algebra: *when do two polynomials have a common factor?* Viewing this question in tropical algebraic geometry leads to a symbolic-numeric algorithm. In particular, we will say that *tropisms give the germs to grow the tentacles of the common amoeba.* The paper is structured in four parts, each part explaining one of the key concepts of the tropical sentence.

Our perspective on tropical algebraic geometry originates from polyhedral homotopies [25], [39], [69] to solve polynomial systems implementing Bernshteĭn's first theorem [4]. Another related

Date: 31 July 2009. This material is based upon work supported by the National Science Foundation under Grant No. 0713018.

†Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, 851 South Morgan (M/C 249), Chicago, IL 60607-7045, USA. *Email:* adrovic@math.uic.edu

‡Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, 851 South Morgan (M/C 249), Chicago, IL 60607-7045, USA. *Email:* jan@math.uic.edu or jan.verschelde@na-net.ornl.gov *URL:* http://www.math.uic.edu/~jan

approach that led to tropical mathematics is idempotent analysis [41]. In [53], a Maple package is presented for a tropical calculus with application to differential boundary value problems.

Related work on our problem concerns the factorization of sparse polynomials via Newton polytopes [1], [16], [19]; approximate factorization [11], [10], [18], [33], [59], and the GCD of polynomials with approximate coefficients [74]. The polynomial absolute factorization is also addressed in [8] and the lectures in [9] offer a very good overview. Criteria based on polytopes for the irreducibility of polynomials date back to Ostrowski [47]. In this paper we restrict our examples to polynomials in two variables and refer to polygons instead of polytopes. The terminology extends to general dimensions and polytopes, see [75].

That two polynomials with approximate coefficients have a common factor is quite an exceptional situation. Therefore it is important to have efficient preprocessing criteria to decide quickly. The preprocessing method we develop in this paper attempts to build a Puiseux expansion starting at a common root at infinity. To determine whether a root at infinity is isolated or not we apply the Newton-Puiseux method, extending the proof outlined by Robert Walker in [71], see also [14], towards Joseph Maurer's general method [42] for space curves. A more algorithmic method than [42] is given in [2] along with an implementation in CoCoA. CASA [23] computes Puiseux series over the rational numbers, see also [57, Appendix A]. General fractional power series solutions are described in [43]. See [30], [31] and [51] for recent symbolic algorithms, and [49], [50] for a symbolic-numeric approach. The complexity for computing Puiseux expansions for plane curves is polynomial [72] in the degrees. As an alternative to Puiseux series, extended Hensel series are discussed in [56], with good numerical convergence reported in [26].

We show that via suitable coordinate transformations, the problem of deciding whether there is a common factor is reduced to univariate root finding, with the univariate polynomials supported on edges of the Newton polygons of the given equations. Also in the computation of the second term of the Puiseux series expansion, we do not need to utilize all coefficients of the given polynomials. In the worst case, the cost of deciding whether there is a factor is a cubic polynomial in the number of monomials of the given polynomials.

Certificates for the existence of a common factor consist of exact and approximate data: the exponents and coefficients of the first two terms of a Puiseux series expansion of the factor at a common root at infinity. The leading exponents of the Puiseux expansion form a so-called *tropism* [42]. The coefficients are numerical solutions of overdetermined systems. If a more explicit form of the common factor is required, more terms in the Puiseux expansion can be computed up to precision needed for the application of sparse interpolation techniques, see [13], [21], [32], and [36].

The `ConvexHull` and `subs` commands of Maple are very valuable in implementing an interactive prototype of the preprocessing algorithm. For explaining the intuition behind the algorithm, we start illustrating amoebas and the tentacles. Once we provide an abstraction for the tentacles we give an outline of the algorithm and sketch its cost. The Maple code served as a prototype for an implementation in PHCpack [67].

Acknowledgements. This paper is based on the talk the second author gave at MICA 2008 – Milestones in Computer Algebra, a conference in honour of Keith Geddes' 60th Birthday, Stonehaven Bay, Trinidad and Tobago, 1-3 May. We thank the organizers of this wonderful and inspiring conference for their invitation. We are grateful to the referees for their comments.

2 Amoebas

Looking at the asymptotics of varieties gives a natural explanation for the Newton polygon. This polygon will provide a first classification of the approximate coefficients of the given polynomials. This means that at first we may ignore coefficients of monomials whose exponents lie in the interior of the Newton polygon.

2.1 Asymptotics of Varieties

Our input data are polynomials in two variables x and y. The set of values for x and y that make the polynomials zero is called a variety. Varieties are the main objects in algebraic geometry. In 1971, G.M. Bergman [3] considered logarithms of varieties. In tropical algebraic geometry, we look at the asymptotics of varieties.

$$\begin{array}{rcccc} \log & : & \mathbb{C}^* \times \mathbb{C}^* & \rightarrow & \mathbb{R} \times \mathbb{R} \\ & & (x, y) & \mapsto & (\log(|x|), \log(|y|)) \end{array} \tag{1}$$

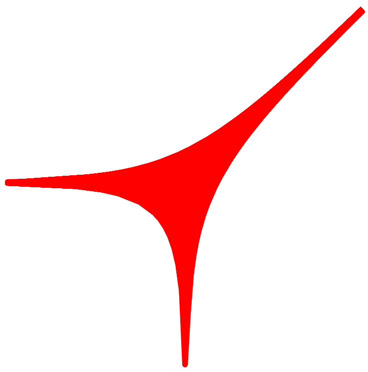
Because the logarithm is undefined at zero, we exclude the coordinate axes restricting the domain of our polynomials to the torus $(\mathbb{C}^*)^2$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Following [20], we arrive at our first new word [70].

Definition 2.1 (Gel'fand, Kapranov, and Zelevinsky 1994) The *amoeba* of a variety is its image under the log map.

Example 2.2 To see what amoebas look like, we use the plotting capabilities of Maple. We use polar coordinates to plot a linear variety:

$$f := \frac{1}{2}x + \frac{1}{5}y - 1 = 0 \qquad A := \left[\ln\left(\left| re^{I\theta} \right| \right), \ln\left(\left| \frac{5}{2}re^{I\theta} - 5 \right| \right) \right]. \tag{2}$$

In Figure 1 we see the result of a Maple plot.



```
> f := 1/2*x + 1/5*y - 1:
> s := solve(f,y):
> L := map(log,map(abs,[x,s])):
> A := subs(x=r*exp(I*theta),L);
> Ap := seq(plot([op(subs(theta=k*Pi/200,A)),
        r=-100..100],thickness=6),k=0..99):
> plots[display](Ap,axes=none);
```

Figure 1: The amoeba of a linear polynomial, with all Maple commands at the right.

2.2 Compactifying Amoebas leads to Newton Polytopes

We compactify the amoeba of $f^{-1}(0)$ by taking lines perpendicular to the tentacles. As each line cuts the plane in half, we keep those halves of the plane where the amoeba lives. The intersection of all half planes defines a polygon. The resulting polygon is the Newton polygon of f. There is a map [60] that sends every point in the variety to the interior of the Newton polygon of the defining polynomial equation.

Example 2.3 (Example 2.2 continued) For the amoeba in Figure 1, its compactification is shown in Figure 2. In Figure 2 we recognize the shape of the triangle, the Newton polygon of a linear polynomial.

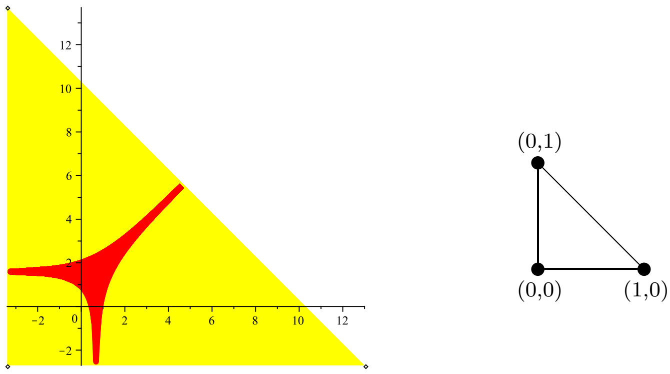


Figure 2: The compactification of the amoeba: the edges of the Newton polygon (displayed at the right) are perpendicular to the tentacles of the amoeba.

This geometric derivation of the Newton polygon coincides with the more formal definition.

Definition 2.4 For $f(x,y) = \sum_{(i,j)\in A} c_{i,j} x^i y^j$, $c_{i,j} \in \mathbb{C}^*$. A is the *support* of f. The convex hull of A is the *Newton polygon*.

The Newton polygon models the sparse structure of a polynomial. Most polynomials arising in practical applications have few monomials with nonzero coefficients and are called *sparse*. The Newton polygon assigns additional significance to the coefficients. Coefficients associated to monomials whose exponents span a vertex of the Newton polygon are more important than coefficients whose exponents lie in the interior of the Newton polygon.

Plotting amoebas is actually computationally quite involved – the use of homotopy continuation methods [67] is suggested in [64]. A computer program to plot amoebas is presented in [38]. See [45], [46], and [48] for more about amoebas. We will see that the asymptotics of the amoebas will lead to a natural reduction of our problem to smaller polynomials *in one variable*.

3 Tentacles

The tentacles of the amoeba stretch out to infinity and are represented by the inner normals, perpendicular to the edges of the Newton polygon.

3.1 Directions of Tentacles towards Infinity

Our problem may be stated as follows: Given two polynomials in two variables with *approximate* complex coefficients, is there a common factor?

Looking at the problem from a tropical point of view, we first have the amoeba of the common factor in mind and we consider its tentacles. Following [70], a tentacle is a rapidly thinning end of the amoeba. More formally, along [44, Remark 9], we consider the closure $\bar{A}$ of the amoeba in the toric variety [12] associated to the Newton polygon of the defining polynomial of the amoeba. Then the tentacles of the amoeba correspond to the intersections of $\bar{A}$ with the edges of the Newton polygon. In the plane, these intersections are isolated points.

The tropical view will lead to solving the problem first at infinity, providing an efficient preprocessing criterion. Figure 3 illustrates the geometric idea of Proposition 3.1.

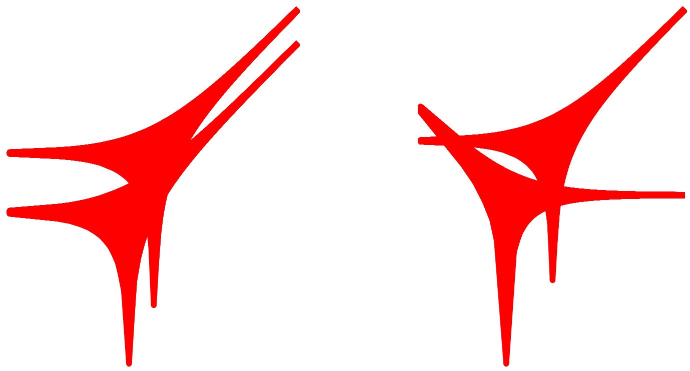


Figure 3: The amoebas of $(\frac{1}{2}x + \frac{1}{5}y + 1)(x + y + 1)$ and $(\frac{1}{2}x + \frac{1}{5}y + 1)(xy + y + \frac{1}{2})$, respectively at the left and right. The amoeba of a product is the union of the amoebas of the factors. Observe the directions of the tentacles.

Proposition 3.1 *Let f and g be two polynomials. If the amoebas of f and g have no tentacle stretching out to infinity in the same direction, then f and g have no common factor.*

Proof. We proceed by contraposition, assuming f and g have a common factor, say r, and we write $f = rf_1$ and $g = rg_1$. The tentacles of the amoebas of f and g will contain the tentacles of the common factor r because of $f^{-1}(0) = r^{-1}(0) \cup f_1^{-1}(0)$ and therefore $A_f = A_r \cup A_{f_1}$, where the amoebas of f, r, and f_1 are denoted respectively by A_f, A_r, and A_{f_1}. Similarly, for g: $A_g = A_r \cup A_{g_1}$. So A_f and A_g contain both A_r and the same intersection points with the edges of the Newton polygons and therefore tentacles stretching out in the same directions. $\square$

Verifying the conditions of Proposition 3.1 seems nontrivial at first. However, we represent the tentacles by inner normals, perpendicular to the lines at infinity corresponding to the edges of the Newton polygons. Because the factor is common to both polynomials, the normals must be common to both polygons. So if there is a factor, there must be at least one pair of edges with the same inner normal vector. Such inner normal vector is a *tropism*, defined below.

3.2 Normal Fans and Tropicalization

The inward pointing normal vectors to the edges represent the tentacles of the amoeba.

Example 3.2 Consider for example

$$f := x^3y + x^2y^3 + x^5y^3 + x^4y^5 + x^2y^7 + x^3y^7. \tag{3}$$

In Figure 4 we show the Newton polygon of f and its normal fan.

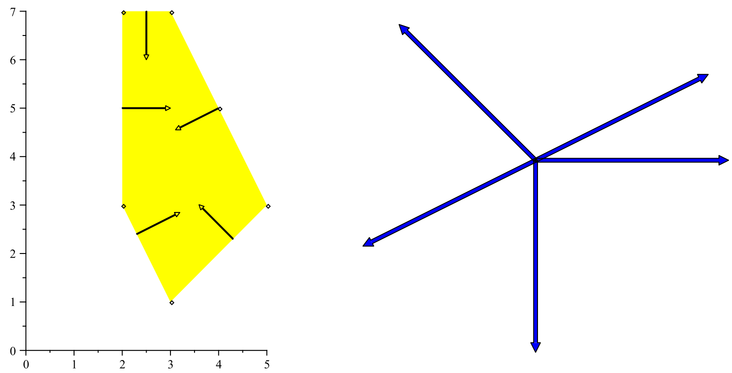


Figure 4: The Newton polygon and its normal fan.

The collection of inner normals to the edges of the Newton polygon forms *a tropicalization* of f, denoted by Trop(f). To formalize this notion, we introduce the following definitions.

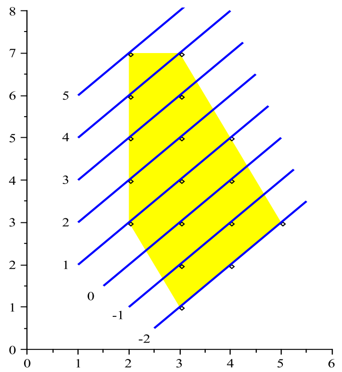
Exponents and direction vectors are related through duality via the inner product.

Definition 3.3 The *inner product* is

$$\begin{array}{cccc} \langle \cdot, \cdot \rangle & \mathbb{Z}^2 \times \mathbb{Z}^2 & \rightarrow & \mathbb{Z} \\ & ((i,j),(u,v)) & \mapsto & iu + jv. \end{array} \tag{4}$$

Given a vector (u,v), $\langle \cdot, (u,v) \rangle$ ranks the points (i,j). For $(u,v) = (1,1)$, we have the usual degree of x^iy^j. So the direction of the tentacles are grading the points in the support.

Example 3.4 (Example 3.2 continued) In Figure 5 we look at the support in the direction $(-1,+1)$ and grade every point of the support using the inner product of its coordinates with $(-1,+1)$.



$$
\begin{aligned}
-1 \times i + (+1) \times j &= +5 \\
-1 \times i + (+1) \times j &= +4 \\
-1 \times i + (+1) \times j &= +3 \\
-1 \times i + (+1) \times j &= +2 \\
-1 \times i + (+1) \times j &= +1 \\
-1 \times i + (+1) \times j &= 0 \\
-1 \times i + (+1) \times j &= -1 \\
-1 \times i + (+1) \times j &= -2
\end{aligned}
$$

Figure 5: Grading the points in the support along $(-1, +1)$.

The degree of $x^i y^j$ in the direction (u, v) is the value of the inner product $\langle (i, j), (u, v) \rangle$. In Maple we compute weighted degrees as follows:

```
Groebner[WeightedDegree](f,[-1,+1],[x,y]);
```

hinting at the connection between Gröbner bases and Newton polytopes [61]. This grading leads to homogeneous coordinates, see [12] and [68].

We arrive at a tropicalization of a polynomial via the normal fan to the Newton polygon of the polynomial.

Definition 3.5 Let P be the Newton polygon of f. The inner product is denoted by $\langle \cdot, \cdot \rangle$. The *normal cone to a vertex* $\mathbf{p}$ *of* P is

$$\{ \ \mathbf{v} \in \mathbb{R}^2 \setminus \{0\} \mid \langle \mathbf{p}, \mathbf{v} \rangle = \min_{\mathbf{q} \in P} \langle \mathbf{q}, \mathbf{v} \rangle \ \}. \tag{5}$$

The *normal cone to an edge spanned by* $\mathbf{p}_1$ *and* $\mathbf{p}_2$ is

$$\{ \ \mathbf{v} \in \mathbb{R}^2 \setminus \{0\} \mid \langle \mathbf{p}_1, \mathbf{v} \rangle = \langle \mathbf{p}_2, \mathbf{v} \rangle = \min_{\mathbf{q} \in P} \langle \mathbf{q}, \mathbf{v} \rangle \ \}. \tag{6}$$

The *normal fan* of P is the collection of all normal cones to vertices and edges of P. Given f, *a tropicalization of* f, denoted by $\mathrm{Trop}(f)$, is a finite collection of inner normals (u, v), its components relatively prime: $\gcd(u, v) = 1$, to the edges of the Newton polygon P of f.

We speak of *a* tropicalization (*a* instead of *the*) because in the general construction of a tropical variety of an ideal [62, §9.4], one often introduces an auxiliary variable t. In our setting, this t does not occur, so our tropicalizations are more restricted. In particular, in [48], the tropicalization f^τ of a Laurent polynomial f with support A is defined as

$$f^\tau(\mathbf{x}) = \max_{\mathbf{a} \in A} \{ \ \log |c_{\mathbf{a}}| + \langle \mathbf{a}, \mathbf{x} \rangle \ \} \quad \text{for} \quad f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, c_{\mathbf{a}} \in \mathbb{C}^*. \tag{7}$$

We prefer min over max because we consider Puiseux series around zero. Ignoring coefficient size: $O(c_{\mathbf{a}}) = 1$ and omitting $\log |c_{\mathbf{a}}|$ from the definition of f^τ, the tropical variety $(f^\tau)^{-1}(0)$ consists of those points $\mathbf{v}$ where at least two of the monomials have the extremal value $\langle \mathbf{a}, \mathbf{v} \rangle$.

4 Tropisms

The tropical view will lead to an efficient preprocessing stage to determine whether two polynomials have a common factor.

4.1 Turning the Varieties in a Particular Direction

The answer to our original question "*Do two polynomials have a common factor?*" first depends on the relative orientations of the Newton polygons. We compute tropicalizations of the polynomials and obtain an efficient preprocessing step independent of the coefficients.

We first want to exclude the situations where there is no common factor, already implied by the Newton polygons in relative general position. This is a direct consequence of Bernshteĭn's second theorem [4]. For completeness, we state this theorem here for Newton polygons.

Theorem 4.1 *Let f and g be two polynomials in x and y. If* $\mathrm{Trop}(f) \cap \mathrm{Trop}(g) = \emptyset$ *then the system $f(x, y) = 0 = g(x, y)$ has no solutions at infinity.*

We will prove Theorem 4.1 later, after Definition 4.6. Now we can make Proposition 3.1 effective:

Proposition 4.2 *If for two polynomials f and g:* $\mathrm{Trop}(f) \cap \mathrm{Trop}(g) = \emptyset$*, then f and g have no common factor.*

Proof. By Theorem 4.1, $\mathrm{Trop}(f) \cap \mathrm{Trop}(g) = \emptyset$ implies there is no common root at infinity. However, if f and g had a common factor, they would have a common root at infinity as well. This common root would then correspond to one of the ends of the tentacles of the amoeba of the common factor as in Proposition 3.1. □

Example 4.3 For our first pair of two random polynomials (each of degree 15), their tropicalizations are shown in Figure 6.

Example 4.4 In our second example we generated a factor of degree 5 and multiplied the factor with two random polynomials f and g of degree 10. A tropicalization of the factor and the two polynomials f and g are shown in Figure 7.

A dictionary definition of a tropism is *the turning of all or part of an organism in a particular direction in response to an external stimulus.* Tropisms were introduced mathematically in 1980 by Joseph Maurer [42] who generalized Puiseux expansions for space curves. We adapt his definition for use to our problem.

Definition 4.5 Let P and Q be Newton polygons of f and g. A *tropism* is an inner normal perpendicular to one edge of P and one edge of Q.

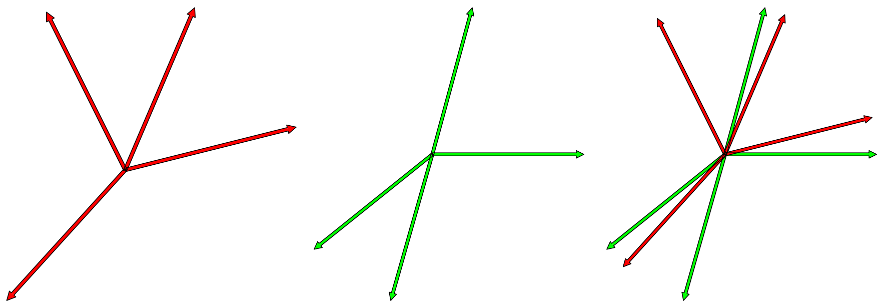


Figure 6: The first two pictures from the left represent the normal fans of two polynomials. By superposition of the fans at the far right we see there are no common directions. Therefore, for all nonzero coefficients, the polynomials can have no common factor.

Using the general terminology of [75], tropisms correspond to the one dimensional cones in the common refinement of the normal fans of the polygons. Note that tropisms in the original sense as used in [42] correspond to leading exponents of actual Puiseux series and that our inner normals may not lead to Puiseux series. Tropisms also occur in singularity theory [37]. In a stricter use of terminology, we would label the inner normals of Definition 4.5 as candidate tropisms or pretropisms. The "pre" of pretropism refers to the tropical prevariety, obtained as the intersection of tropical hypersurfaces [54].

4.2 Certificates for Numerical Computations

Tropisms are important because they give a first exact certificate for the existence of a common factor. Selecting those monomials which span the edges picked out by the tropism defines a polynomial system which admits a solution in $(\mathbb{C}^*)^2$.

Definition 4.6 Let (u,v) be a direction vector. Consider $f = \sum_{(i,j)\in A} c_{i,j}x^iy^j$. The *initial form of f in the direction (u,v)* is

$$\mathrm{in}_{(u,v)}(f) = \sum_{\substack{(i,j)\in A \\ \langle (i,j),(u,v)\rangle = m}} c_{i,j}x^iy^j, \qquad (8)$$

where $m = \min\{\ \langle (i,j),(u,v)\rangle \mid (i,j)\in A\ \}$.

The direction (u,v) is the normal vector to the line $ui + vj = m$ which contains the edge of the Newton polygon of f. This edge is the Newton polygon of $\mathrm{in}_{(u,v)}(f)$.

The terminology of initial forms corresponds to the Gröbner basics [61]. In [55], $\mathrm{in}_{(u,v)}(f)$ is called an initial term polynomial. We call a tuple of initial forms *an initial form system.*

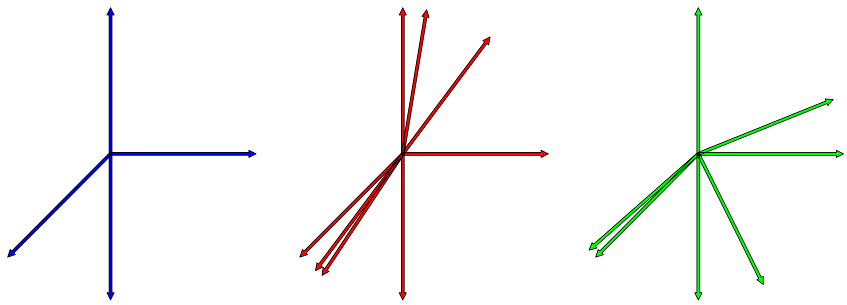


Figure 7: The normal fan at the left is the normal fan of the factor common to two polynomials f and g. The normal fans of f and g are displayed in the middle and at the right. We recognize the fan at the left as a part of the other fans.

Initial form systems are called truncated systems in [7] and [34]. At this point we can show how Theorem 4.1 is a direct consequence of [4, Theorem B].

Proof of Theorem 4.1. Rephrasing part (a) of [4, Theorem B], using our notations and restricting to two polynomials f and g in x and y: If the system defined by the equations $\mathrm{in}_{\mathbf{v}}(f)(x,y) = 0$ and $\mathrm{in}_{\mathbf{v}}(g)(x,y) = 0$ does not have any roots in $(\mathbb{C}^*)^2$ for any $\mathbf{v} \neq (0,0)$, then all roots of the system defined by $f(x,y) = 0$ and $g(x,y) = 0$ are isolated and their number equals the mixed volume of the polygons spanned by the supports of f and g. The condition $\mathrm{Trop}(f) \cap \mathrm{Trop}(g) = \emptyset$ implies there is no $\mathbf{v}$ so that $\mathrm{in}_{\mathbf{v}}(f)$ and $\mathrm{in}_{\mathbf{v}}(g)$ have each at least two monomials. Equivalently, for all $\mathbf{v} \neq (0,0)$, $\mathrm{in}_{\mathbf{v}}(f)$ or $\mathrm{in}_{\mathbf{v}}(g)$ (possibly both for general $\mathbf{v}$, but at least one of them for particular choices of $\mathbf{v}$) consist only of one monomial. Therefore the system defined by the equations $\mathrm{in}_{\mathbf{v}}(f)(x,y) = 0$ and $\mathrm{in}_{\mathbf{v}}(g)(x,y) = 0$ does not have any roots in $(\mathbb{C}^*)^2$. Hence, the system defined by $f(x,y) = 0$ and $g(x,y) = 0$ has no roots at infinity. □

Example 4.7 (Example 4.4 continued) For the common factor r, the polynomials f and g generated using Maple's `randpoly` were

$$r := 2xy + x^2y + 9xy^2 + 7x^3y + x^4y + 9x^3y^2, \tag{9}$$

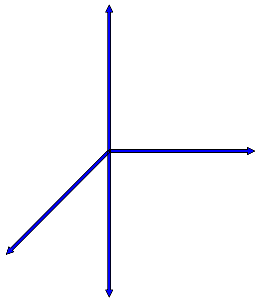
$$f := r\left(6x^{10} + 6x^6y^3 + 5x^4y + 3x^3y^5 + 5y^4 + 5y^5\right), \tag{10}$$

$$g := r\left(2x^{13} + 5x^9 + x^6y^3 + 8x^6y^8 + 6x^2 + 5y^5\right). \tag{11}$$

Because we exclude the coordinate axes, the factor xy of r is considered trivial and is not reported as a separate factor. In Figure 8 we show the initial forms of the two polynomials defined by the tropism $(1,0)$.

Because the tropism is a standard basis vector (1,0), the initial form system it determines consists of two polynomials in one variable after canceling monomial factors:

$$\begin{cases} \mathrm{in}_{(1,0)}(f) = x\left(5y^5(y+1)(2+9y)\right) = 0 \\ \mathrm{in}_{(1,0)}(g) = x\left(5y^5(2+9y)\right) = 0 \end{cases} \tag{12}$$



Take $(1,0)$ as one of the 4 directions:

$\mathrm{in}_{(1,0)}(r) = 2xy + 9xy^2$

Initial forms of f and g:

$\mathrm{in}_{(1,0)}(f) = 55xy^6 + 10xy^5 + 45xy^7 = \mathrm{in}_{(1,0)}(r)(5y^4 + 5y^5)$

$\mathrm{in}_{(1,0)}(g) = 10xy^6 + 45xy^7 = \mathrm{in}_{(1,0}(r)(5y^5)$

Figure 8: The normal fan of the common factor and the initial form systems corresponding to the direction (1,0).

and then $y = -2/9$ represents the common root at infinity. Note that the x-coordinate for this root at infinity equals zero. Excluding coordinate axes, $x = 0$ is considered at infinity.

In general the common root at infinity will be an approximate root and with α-theory [5] we can bound the radius of convergence for Newton's method. For polynomials p in one variable, the gamma function $\gamma(p,z)$ can be computed in a straightforward manner for any regular root z, as the maximum of $\left|\frac{p^{(k)}(z)}{k!p'(z)}\right|^{1/(k-1)}$, for all k ranging from 2 to the degree of p. Then a lower bound for the radius of convergence for Newton's method is $(3-\sqrt{7})/(2\gamma(p,z))$. In [58], this notion of approximate zeroes was extended to include approximate functions, when the Newton operator cannot be evaluated exactly. An alternative to this approach is take one root of the first polynomial and compute how much the coefficients of the second polynomial must change for it to have the same root [24]. In addition to the first certificate, the exact tropism, the common root at infinity is the second approximate certificate for a potential common factor of two polynomials.

For general tropisms, not equal to basis vectors, we perform unimodular transformations in the space of the exponents to reduce the initial form system to a system of two polynomial in one variable. In [7], the coordinate transformations resulting from those unimodular transformations are called power transformations and they power up the field of "Power Geometry".

Example 4.8 (Example 4.4 continued) Investigating the direction $(-1,-1)$:

$$\left\{ \begin{array}{rcl} \mathrm{in}_{(-1,-1)}(f) & = & 54x^{13}y^2 + 6x^{14}y = \left(x^4y + 9x^3y^2\right) 6x^{10} \\ \mathrm{in}_{(-1,-1)}(g) & = & 72x^9y^{10} + 8x^{10}y^9 = \left(x^4y + 9x^3y^2\right) 8x^6y^8 \end{array} \right. \qquad (13)$$

Using the unimodular matrix $M = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$, $\gcd(-1,-1) = (-1)(-1) + 0(-1) = 1$, we will change coordinates.

Definition 4.9 For a tropism (u,v) normalized so the greatest common divisor $\gcd(u,v) = 1$, the unimodular matrix M

$$M = \begin{bmatrix} u & v \\ -l & k \end{bmatrix}, \quad \gcd(u,v) = 1 = ku + lv = \det(M) \qquad (14)$$

defines *the unimodular coordinate transformation* $x = X^u Y^{-l}$ and $y = X^v Y^k$.

Note that for a monomial $x^a y^b$, the coordinate transformation yields

$$(X^u Y^{-l})^a (X^v Y^k)^b = X^{au+bv} Y^{-la+kb} = X^{\langle (a,b),(u,v) \rangle} Y^{-la+kb}, \qquad (15)$$

so after the coordinate transformation, the monomials in the initial forms all have the same minimal degree in X.

Example 4.10 (Example 4.8 continued) We perform the change of coordinates:

$$\left\{ \begin{array}{lcl} \mathrm{in}_{(-1,-1)}(f)(x = X^{-1}, y = X^{-1}Y^{-1}) & = & (54Y + 6)/(X^{15}Y^2) \\ \mathrm{in}_{(-1,-1)}(g)(x = X^{-1}, y = X^{-1}Y^{-1}) & = & (72 + 8Y)/(X^{19}Y^{10}) \end{array} \right. \qquad (16)$$

This change of coordinates reduces the initial form system to a system of two polynomials in one variable. For the example, $Y = -1/9$ represents the common root at infinity. Going back to the original coordinates:

$$\left\{ \begin{array}{l} X = t \\ Y = -1/9 \end{array} \right. \quad \left(\begin{array}{l} x = X^{-1} \\ y = X^{-1}Y^{-1} \end{array} \right) \quad \Rightarrow \quad \left\{ \begin{array}{l} x = t^{-1} \\ y = -9t^{-1}. \end{array} \right. \qquad (17)$$

As t goes to 0 we have indeed a root going off to infinity.

For every tentacle of the common factor we can associate a degree as follows. Considering again the common factor r from (9), the amoeba for r has four tentacles, see Figure 8, reflected by its tropicalization

$$\mathrm{Trop}(r) = \{ \ (1,0), (0,1), (-1,-1), (0,-1) \ \}. \qquad (18)$$

In Table 1 we list the degrees associated to each tentacle of the common factor. We count the number of nonzero solutions of the initial forms, after proper unimodular coordinate transformation. To make the correspondence with the usual degree, observe that we ignore monomial factors.

(u,v)	$\mathrm{in}_{(u,v)}(r)$	degree
$(1,0)$	$2xy + 9xy^2$	1
$(0,1)$	$2xy + x^2y + 7x^3y + x^4y$	3
$(-1,-1)$	$x^4y + 9x^3y^2$	1
$(0,-1)$	$9xy^2 + 9x^3y^2$	2

Table 1: Degrees associated to each vector in $\mathrm{Trop}(r)$.

Summing the vectors in the first column of Table 1 yields zero. In general, the inner normals to the edges of the Newton polygon satisfy what is known as the balancing condition [54]: with every vector $\mathbf{v}_k$ of the tropicalization one can assign a multiplicy m_k so that all $m_k \times \mathbf{v_k}$'s sum up to zero. This balancing is used in [22] to factor tropical polynomials. Thus we do not need all tropisms to represent a factor. In the extreme case of a binomial factor, e.g.: $x - y$, we will find the tropisms $(1,1)$ and $(-1,-1)$, corresponding to $(x = t, y = t)$ and $(x = t^{-1}, y = t^{-1})$ respectively.

4.3 A Preprocessing Algorithm and its Cost

That two polynomials with approximate coefficients have a common factor does not happen that often. Therefore, it is important to be able to decide quickly in case there is no common factor. The stages in a preprocessing algorithm are sketched in Figure 9.

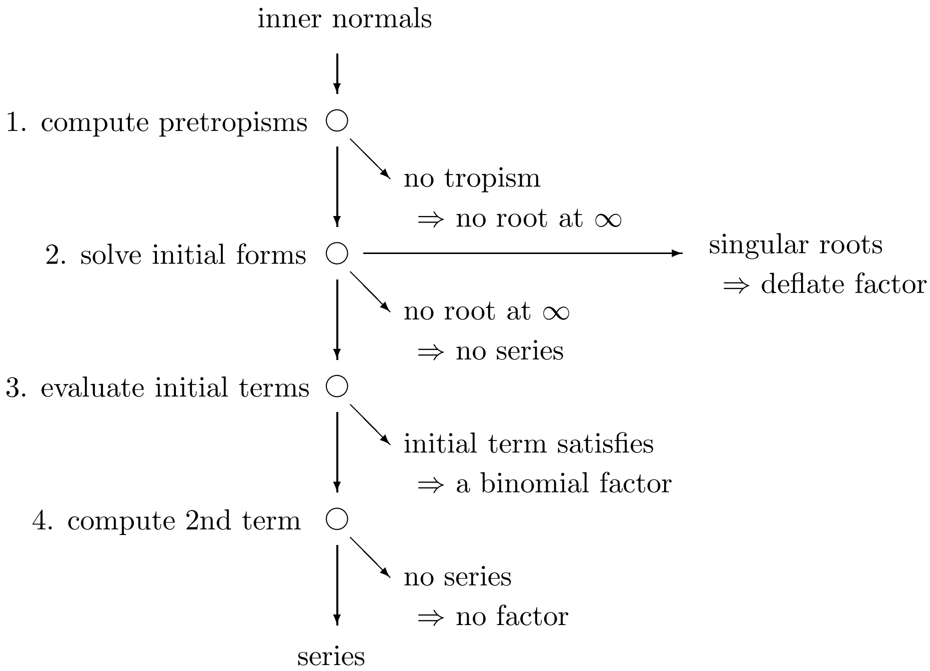


Figure 9: A staggered approach for a regular common factor of two polynomials in two variables.

In Figure 9 we distinguish four computational steps. We will address the cost of the first two steps in the following propositions.

Proposition 4.11 *Let f and g be two polynomials given by respectively n and m monomials. The cost of computing tropisms* $\mathrm{Trop}(f) \cap \mathrm{Trop}(g)$ *is* $O(n \log(n)) + O(m \log(m))$.

Proof. It takes $O(n \log(n))$ operations for computing a tropicalization $\mathrm{Trop}(f)$ because computing the convex hull of a set of n points amounts to sorting the points in the support. Likewise, computing $\mathrm{Trop}(g)$ takes $O(m \log(n))$ operations. Merging sorted lists of normals to find the tropisms in $\mathrm{Trop}(f) \cap \mathrm{Trop}(g)$ takes linear time in the length of the lists. □

This preprocessing step has the lowest complexity and as the algorithm operates only on the exponents the outcome is exact. The absence of tropisms is an exact certificate that there is no common factor, for any nonzero choice of the coefficients of the polynomials.

In case we have tropisms, we solve initial form systems. The cost of the second preprocessing stage is as follows.

Proposition 4.12 *Let f and g be two polynomials given by respectively n and m monomials. For every tropism $t \in \mathrm{Trop}(f) \cap \mathrm{Trop}(g)$ it takes at most $O((n+m)^3)$ operations to find a common solution in $(\mathbb{C}^*)^2$ to the initial form system defined by $\mathbf{v}$.*

Proof. For a tropism $\mathbf{v}$, we solve the initial form system. In particular, an initial root $\mathbf{z}$ satisfies

$$\left\{ \begin{array}{l} \mathrm{in}_{\mathbf{v}}(f)(\mathbf{z}) = 0 \\ \mathrm{in}_{\mathbf{v}}(g)(\mathbf{z}) = 0 \end{array} \right. \quad \mathbf{z} \in (\mathbb{C}^*)^2. \tag{19}$$

We perform a unimodular transformation so the tropism we consider is a unit vector, (1,0) or (0,1). This implies that the two equations in the initial form system are defined by two polynomials in one variable. To decide whether two polynomials in one variable admit a common solution we determine the rank of the Sylvester matrix. Using singular value decomposition, the cost of this rank determination is cubic in the size of the matrix. For rank deficient matrices, the singular vectors give the coefficients of the common factor. The roots of this common factor are the eigenvalues of a companion matrix. The cost of methods to compute eigenvalues is also cubic in the dimension of the matrix. □

Even as the cost estimates in the propositions are conservative, they give a good polynomial complexity. Actually, in the best case, the initial forms are supported on two points only and instead of a rank determination, we can just take primitive roots. The cost estimates of Proposition 4.12 cover the very worst situation where the Newton polygons are triangles and one of the edges contains all exponent vectors except for one.

For numerical calculations, it is important to note that at this preprocessing stage, only the coefficients at the edges are involved. If the coefficients are badly scaled, then coefficients with monomials in the interior of the Newton polygons will not cause difficulties at this stage.

For the complexity in the proof of the second proposition we used the ubiquitous singular value decomposition but for practical purposes rank revealing algorithms [40] have a lower cost. The accurate location of the root of the initial form systems may look complication in case this root is multiple. However, because the initial form systems consists of univariate equations, the methods of [73] will give satisfactory answers.

The "deflate factor" of Figure 9 means that we would work with the derivatives of f and g in case a multiple initial root is found. For example, suppose the common factor r occurs with multiplicity two in f: $f = r^2 f_1$. Then, by $\frac{\partial f}{\partial x} = 2r\frac{\partial r}{\partial x} f_1 + r^2 \frac{\partial f_1}{\partial x}$ we see that r is a regular factor of the partial derivatives of f.

Before we move to the computation of the second term of a Puiseux series of a common factor, we point at the third stage of Figure 9, that deals with cases when no second term exists, i.e.: when the common factor has only two monomials with nonzero coefficient. We call such factor a binomial factor. If the evaluation of the initial term in the polynomials f and g turns out to be zero, then we have a binomial factor. In the other direction, if there is a binomial factor, then after a unimodular transformation it has the form $X^k(c_0 + c_l Y^l)$ and is therefore satisfied by $(X = t, Y = z)$, for some zero of $c_0 + c_l Y^l = 0$.

5 Germs

Once we have a tropisms and an initial root at infinity, we start growing the Puiseux series for the common factor.

5.1 How the Amoeba Grows from Infinity

We use the roots at infinity to grow the tentacles of the common factor. But first we must decide whether the roots at infinity are isolated or not. We first define the representation of the common factor.

Definition 5.1 Consider the curve defined by $r(x, y) = 0$. Except for eventual monomial factors, r has no multiple factors. In canonical form for the tropism $(1, 0)$, a *Puiseux series for* $(1, 0)$ has the form

$$\left\{ \begin{array}{lcl} X & = & t \\ Y & = & c_0 + c_1 t^w (1 + O(t)), \quad c_0, c_1 \in \mathbb{C}^*, w \in \mathbb{N}, w > 0. \end{array} \right. \tag{20}$$

For a general tropism $(u, v) \in \mathbb{Z}^2$, with $\gcd(u, v) = ku + lv = 1$, a *Puiseux series for* (u, v) has the form

$$\left\{ \begin{array}{lclll} x & = & t^u \left(c_0 + c_1 t^w (1 + O(t))\right)^{-l} & c_0, c_1 \in \mathbb{C}^* & x = X^u Y^{-l} \\ y & = & t^v \left(c_0 + c_1 t^w (1 + O(t))\right)^{k} & w \in \mathbb{N}, w > 0 & y = X^v Y^k \end{array} \right. \tag{21}$$

Observe the unimodular transformation, going from the original coordinates (x, y) to (X, Y), used to find c_0 as a solution of the initial form system

$$\left\{ \begin{array}{l} \text{in}_{(1,0)}(f)(t, c_0) = 0 \\ \text{in}_{(1,0)}(g)(t, c_0) = 0 \end{array} \right. \tag{22}$$

where the initial forms are taken from the equations f and g which define the common factor r. In this section we will consider the calculation of the second term $c_1 t^w$ of the series.

Example 5.2 (Example 4.4 continued) We extend the solution at infinity, defined by the initial form system for the first tropism (1,0). Because the tropism is a standard basis vector, the Maple command `sort({ f , g }, plex, ascending)` will show that the leading terms of the polynomials f and g are indeed $\text{in}_{(1,0)}(f)$ and $\text{in}_{(1,0)}(g)$:

$$\left\{ \begin{array}{l} f = 10xy^5 + 45xy^7 + 55xy^6 + x^2(\text{ 30 other terms }) \\ g = 45xy^7 + 10xy^6 + x^2(\text{ 34 other terms }) \end{array} \right. \tag{23}$$

Let $f_1 = f/x$ and $g_1 = g/x$, then $z = -2/9$ is solution at infinity.

$$\left\{ \begin{array}{l} x = t^1 \\ y = -\frac{2}{9} t^0 + Ct(1 + O(t)), \quad c \in \mathbb{C}^*. \end{array} \right. \tag{24}$$

A nonzero value for C will give the third certificate for a common factor. Useful Maple commands to compute the power series are

```
zt := x = t, y = -2/9 + C*t;
f1z := subs(zt,f1): g1z := subs(zt,g1):
c1 := coeff(f1z,t,1); c2 := coeff(g1z,t,1);
```

The constraints on the coefficient C we obtain are

$$\left\{ \begin{array}{l} c1 = -\frac{1120}{531441} - \frac{1120}{59049}C = 0 \\ c2 = -\frac{320}{59049} - \frac{320}{531441}C = 0 \end{array} \right. \tag{25}$$

Notice that the second coefficient C of the Puiseux series expansion again must satisfy an overdetermined system. Solving both equations for C gives $C = -1/9$.

$$\left\{ \begin{array}{l} x = t \\ y = -\frac{2}{9} - \frac{1}{9}t(1 + O(t)). \end{array} \right. \tag{26}$$

Substituting $x = t, y = -2/9 - t/9$ into f_1 and g_1 gives $O(t^2)$. The second term of the Puiseux series is the third and last certificate for a common factor.

In general, the next term in the Puiseux series expansion might have a degree higher than one, or there might not exist a second term at all in case the solution at infinity is isolated. There is an explicit condition on the exponent of the second term in the Puiseux series expansion as in Proposition 5.3.

Proposition 5.3 *Given are two polynomials f and g in X and Y, after a unimodular coordinate transformation and a multiplication or division by a monomial so f and g have the form*

$$\left\{ \begin{array}{ll} f(X,Y) = p(Y) + P(X,Y), & p(Y) = \mathrm{in}_{(1,0)}(f)(X,Y), \\ g(X,Y) = q(Y) + Q(X,Y), & q(Y) = \mathrm{in}_{(1,0)}(g)(X,Y). \end{array} \right. \tag{27}$$

By the given form of f and g, the initial forms p and q are polynomials in Y with nonzero constant term. Moreover, all terms in the remainder polynomials P and Q have a positive power in X. Let $c_0 \neq 0$:

$$\left\{ \begin{array}{lll} p(c_0) = 0, & p'(c_0) \neq 0, & f(t, c_0) \neq 0 \\ q(c_0) = 0, & q'(c_0) \neq 0, & g(t, c_0) \neq 0 \end{array} \right. \quad p' = \frac{\partial p}{\partial Y}, q' = \frac{\partial q}{\partial Y}. \tag{28}$$

Let $P_k \in \mathbb{C} \setminus \{0\}$: $P(X, c_0) = P_k X^k(1 + O(X))$ and $Q_l \in \mathbb{C} \setminus \{0\}$: $Q(X, c_0) = Q_l X^l(1 + O(X))$. If $k = l$ and $Q_k p'(c_0) - P_k q'(c_0) = 0$, then $c_1 = -P_k/p'(c_0) = -Q_k/q'(c_0)$ is the coefficient of the second term in $(X = t, Y = c_0 + c_1 t^k)$, the leading part of a Puiseux series expansion of a regular common factor of f and g. If $k \neq l$ or $Q_k p'(c_0) - P_k q'(c_0) \neq 0$, then f and g have no common factor with expansion starting at $(X = t, Y = c_0)$.

Proof. Let us consider the effect of substituting $X = t, Y = c_0 + c_1 t^w$ into f and g, using the value for the initial root c_0 and treating the second coefficient c_1 and the exponent w as unknowns. We may write $p(Y)$ as

$$p(Y) = \alpha_1(Y - c_0)(Y - \alpha_2)(Y - \alpha_3)\cdots(Y - \alpha_d), \quad d = \deg(p), \alpha_i \in \mathbb{C}, i = 1, 2, 3, \ldots, d. \tag{29}$$

Because c_0 is a regular root of the initial forms: $p'(c_0) \neq 0$ and $c_0 \neq \alpha_i$, $i = 2, 3, \ldots, d$. Then:

$$\begin{aligned} p(Y = c_0 + c_1 t^w) &= \alpha_1(c_1 t^w)(c_0 + c_1 t^w - \alpha_2)(c_0 + c_1 t^w - \alpha_3) \cdots (c_0 + c_1 t^w - \alpha_d) &(30)\\ &= c_1 t^w \alpha_1 (c_0 - \alpha_2)(c_0 - \alpha_3) \cdots (c_0 - \alpha_d)(1 + O(t^w)) &(31)\\ &= c_1 t^w p'(c_0)(1 + O(t^w)). &(32) \end{aligned}$$

Similarly: $q(Y = c_0 + c_1 t^w) = c_1 t^w q'(c_0)(1 + O(t^w))$.

Substitution of $X = t$ and $Y = c_0 + c_1 t^w$ into $P(X, Y)$ leads to $P_k t^k(1 + O(t))$ for a nonzero constant P_k. Observe that the lowest power of t does not involve c_1, but only depends on c_0. If the constant P_k were zero, then this would imply $P(t, c_0) = 0$ for all t and also $f(t, c_0) = 0$, contradicting the assumption $f(t, c_0) \neq 0$. Note that $f(t, c_0) = 0$ occurs in case the common factor is binomial, i.e.: consists only of two monomials with nonzero coefficients.

The result of substituting $X = t, Y = c_0 + c_1 t^w$ into f and g is then

$$\left\{ \begin{array}{l} f(X = t, Y = c_0 + c_1 t^w) = c_1 t^w p'(c_0)(1 + O(t^w)) + P_k t^k (1 + O(t)) = 0 \\ g(X = t, Y = c_0 + c_1 t^w) = c_1 t^w q'(c_0)(1 + O(t^w)) + Q_l t^l (1 + O(t)) = 0. \end{array} \right. \quad (33)$$

For the dominant terms to vanish, we must have $w = k = l$ and solve

$$\left[\begin{array}{cc} p'(c_0) & P_k \\ q'(c_0) & Q_k \end{array} \right] \left[\begin{array}{c} c_1 \\ 1 \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \end{array} \right]. \quad (34)$$

For this linear system in c_1 to have the nonzero solution $c_1 = -P_k/p'(c_0) = -Q_k/q'(c_0)$ the determinant $Q_k p'(c_0) - P_k q'(c_0)$ must equal zero.

To prove the second if statement of the proposition, we first observe that if $Q_k p'(c_0) - P_k q'(c_0) \neq 0$, the linear system in c_1 has no solution and hence there is no Puiseux series expansion of a common factor starting at $(X = t, Y = c_0)$. Now we consider the case $k \neq l$. If $k < l$, then the determinant of the linear system in c_1 equals $-P_k q'(c_0) \neq 0$. Otherwise, for $k > l$, the determinant is $Q_l p'(c_0) \neq 0$. Therefore if $k \neq l$, no solution for c_1 exists and there is also no Puiseux series expansion. □

The two key assumptions of Proposition 5.3 are covered by the earlier stages in the preprocessing algorithm outlined in Figure 9. In case the condition of Proposition 5.3 is satisfied and the linear system admits a nonzero solution for c_1, then the exponent w and coefficient c_1 constitute respectively an exact and an approximate certificate for the existence of a common factor for the two polynomials f and g.

5.2 Regions of Convergence of Puiseux Series

We will consider the convergence of Puiseux series only for series in their canonical form, for the tropism $(1, 0)$. Via a unimodular coordinate transformation, Puiseux series for any tropism can be brought into this canonical form: $(X = t, Y = c_0 + c_1 t^w (1 + O(t)))$, for $c_0, c_1 \in \mathbb{C}^*$ and $w \in \mathbb{N}, w > 0$. We use capital letters X and Y in the given polynomials f and g to denote the effect of the coordinate changes.

To verify whether the second term $c_1 t^w$ is valid we substitute $(X = t, Y = c_0 + c_1 t^w)$ into f and g, ignoring terms of order $O(t^{w+1})$ and higher, and compare the lowest power in t of the result of these substitutions to the lowest powers of t respectively in $f(X = t, Y = c_0)$ and $g(X = t, Y = c_0)$. Note that, since c_0 and c_1 are approximate numbers, we disregard in the result of this substitution terms with coefficients of magnitude less than a certain tolerance, relative to the accuracy of the approximations for c_0 and c_1. In cases when the common factor is as simple as $x + y + 1$, the series $(X = t, Y = -1 - t)$ will of course leave no terms in t after substitution into f and g.

For common factors for which the second term does not complete the series, we formalize the verification by substitution as follows. Let $(X = t, Y = c_0 + c_1 t^w)$ be the start of a Puiseux series in canonical form with w and c_1 satisfiying all the conditions of Proposition 5.3 for a regular common factor of f and g. Then the following holds:

$$\left\{ \begin{array}{ll} f(X = t, Y = c_0) = O(t^{m_1}), & m_1 > 0, \\ g(X = t, Y = c_0) = O(t^{m_2}), & m_2 > 0, \end{array} \right. \tag{35}$$

and

$$\left\{ \begin{array}{ll} f(X = t, Y = c_0 + c_1 t^w) = O(t^{m_1 + k_1}), & k_1 > 0, \\ g(X = t, Y = c_0 + c_1 t^w) = O(t^{m_2 + k_2}), & k_2 > 0. \end{array} \right. \tag{36}$$

This property follows from the construction of w and c_1 in the proof of Proposition 5.3.

The equations (35) and (36) indicate symbolically to what extent the values for X and Y obtained from the start of a Puiseux series are equivalent to points sampled from the curve defined by the common factor of f and g. The powers of t obtained by substitution constitute an algebraic tolerance on the common factor. Numerically, we have a disk centered at the point $(0, c_0)$ in $\mathbb{C}^2$ of sufficiently small but positive radius where we may predict the value of points on the curve defined by the common factor of f and g.

The computed $(X = t, Y = c_0 + c_1 t^w)$ can serve in the predictor-corrector method to sample points from the common curve. These sampled points are then useful to compute additional terms in the Puiseux series, or to directly apply sparse interpolation techniques to determine the support and coefficients of the common factor.

The unimodular coordinate transformations play a very important role also in the accurate evaluation of polynomials [15]. As the size of arguments of the polynomial functions grows, and as the direction of the growth points along the direction of a tentacle of the amoeba, monomials on the faces perpendicular to that direction become dominant. A weighted projective transformation as in [68] will rescale the problem of evaluating a high degree polynomial with approximate coefficients near a root.

We normalized the tropisms (u, v) requiring $\gcd(u, v) = 1$. Multiples of (u, v) lead to equivalent Puiseux series. As we consider Puiseux series as solutions of $f(x, y) = 0$ and $g(x, y) = 0$, we may as well consider f and g to have series for coefficients (like the input of a tropicalization). We apply the following definition to f and g:

Definition 5.4 Let p be a polynomial in x and y with coefficients as series in t, converging in some neighborhood U. Then *the germ of* p is $V(p) = \{ \ (x(t), y(t)) \in U \mid p(x(t), y(t)) = 0 \ \}$.

For more on germs in the literature we refer to [14], [17], and [35].

6 Implementation Aspects

For efficient implementation of the algorithm, the data structures used to represent the polynomials consist of a list of exponent vectors and a coefficient table. More precisely, to represent a polynomial f denoted as

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \quad c_{\mathbf{a}} \in \mathbb{C}^*, \mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \tag{37}$$

we use a list to represent the support A and a lookup table $C[A]$ for the coefficients. The indices of the lookup table C_A are the exponent vectors $\mathbf{a} \in A$. In Maple's index notation: $C[\mathbf{a}] = c_{\mathbf{a}}$.

Separating the support from the coefficient allows an efficient execution of change of monomial orders. If $n = \#A$, then monomial orders on f are stored via permutations of the first n natural numbers. The separation also gives an efficient way to change coordinates, i.e.: we apply the unimodular coordinate transformation only on A. For a unimodular matrix M:

$$MA = \{ \ M\mathbf{a} \mid \mathbf{a} \in A \ \}. \tag{38}$$

Abusing notation, for $z \in \mathbb{C}^*$: Mz denotes the value for Y after applying the coordinate transformation as in Definition 4.9.

The input polynomials f and g with respective supports A_f and A_g are then represented by two tuples: $(A_f, C[A_f])$ and $(A_g, C[A_g])$. The preprocessing algorithms consists of two stages. In the first stage, Algorithm 6.1 computes the tropisms and the roots of the corresponding initial form systems. If the sets of roots are not empty, the exponent and coefficients of the second term in the Puiseux expansions are computed by Algorithm 6.2 in the second stage. We define the specifications of the algorithms below.

Algorithm 6.1 Tropisms and Initial Roots

Input : $(A_f, C[A_f])$ and $(A_g, C[A_g])$.

Output : $T = \{ \ (u,v) \in \mathbb{Z}^2 \setminus (0,0) \mid (u,v) \ \text{ is tropism} \ \}$,
$R[T] = \{ \ \{ \ z \in \mathbb{C}^* \mid \text{in}_{(u,v)}(f)(Mz) = 0, \text{in}_{(u,v)}(g)(Mz) = 0 \ \} \mid (u,v) \in T \ \}$.

Every tropism in T defines a set of roots (possibly empty) of the corresponding initial form system, after application of the unimodular coordinate transformation M. The cost of Algorithm 6.1 is estimated by Proposition 4.11 and Proposition 4.12.

Algorithm 6.2 Second Term of Puiseux Expansion

Input : $(A_f, C[A_f])$, $(A_g, C[A_g])$, T, and $R[T]$.

Output : $W[R[T]] = \{ \ (c,w) \in \mathbb{C}^* \times \mathbb{N}^+ \mid z \in Z \in R[T] \ \}$.

The elements of the set $W[R[T]]$ define the second term of the Puiseux series expansion. In particular, for every $(c,w) \in W[R[T]]$:

$$\begin{cases} X = t^1 \\ Y = z + ct^w \end{cases} \tag{39}$$

where (X, Y) are the new coordinates after applying the transformation of Definition 4.9. Conditions on the existence of the exponent w are given in Proposition 5.3.

The Maple code served well to prototype an implementation in PHCpack [67], release 2.3.48, making the code to find a common factor of two Laurent polynomials available to the user via `phc -f`.

7 Conclusions and Extensions

Like Maple, tropical algebraic geometry is language. Sentences like *tropisms give the germs to grow the tentacles of the common amoeba* express efficient preprocessing stages to detect and compute common factors of two polynomials with approximate coefficients. In this paper we outline a symbolic-numeric algorithm to compute Puiseux series of a common factor of two polynomials. Seeing the problem as a system of two polynomial equations in two variables, the algorithm is a polyhedral method to find algebraic curves. Connections with numerical algebraic geometry are described in [66].

Among the extensions we consider for future developments are algorithms to handle singularities numerically and polyhedral methods for space curves.

References

[1] F.K. Abu Salem. An efficient sparse adaptation of the polytope method over $\mathbb{F}_p$ and a record-high binary bivariate factorisation. *J. Symbolic Computation*, 43(5):311–341, 2008.

[2] M. Alonso, T. Mora, G. Niesi, and M. Raimondo. Local parametrization of space curves at singular points. In B. Falcidieno, I. Herman, and C. Pienovi, editors, *Computer Graphics and Mathematics*, pages 61–90. Springer-Verlag, 1992.

[3] G.M. Bergman. The logarithmic limit-set of an algebraic variety. *Transactions of the American Mathematical Society*, 157:459–469, 1971.

[4] D.N. Bernshteǐn. The number of roots of a system of equations. *Functional Anal. Appl.*, 9(3):183–185, 1975. Translated from *Funktsional. Anal. i Prilozhen.*, 9(3):1–4,1975.

[5] L. Blum, F. Cucker, M. Shub, and S. Smale. *Complexity and Real Computation.* Springer-Verlag, 1998.

[6] T. Bogart, A.N. Jensen, D. Speyer, B. Sturmfels, and R.R. Thomas. Computing tropical varieties. *J. Symbolic Computation*, 42(1):54–73, 2007.

[7] A.D. Bruno. *Power Geometry in Algebraic and Differential Equations*, volume 57 of *North-Holland Mathematical Library.* Elsevier, 2000.

[8] G. Chèze. Absolute polynomial factorization in two variables and the knapsack problem. In J. Gutierrez, editor, *Proceedings of the 2004 International Symposium on Symbolic and Algebraic Computation (ISSAC 2004)*, pages 87–94. ACM, 2004.

[9] G. Chèze and A. Galligo. Four lectures on polynomial absolute factorization. In *Solving Polynomial Equations. Foundations, Algorithms and Applications*, volume 14 of *Algorithms and Computation in Mathematics*, pages 339–394. Springer-Verlag, 2005.

[10] R.M. Corless, A. Galligo, I.S. Kotsireas, and S.M. Watt. A geometric-numeric algorithm for factoring multivariate polynomials. In T. Mora, editor, *Proceedings of the 2002 International Symposium on Symbolic and Algebraic Computation (ISSAC 2002)*, pages 37–45. ACM, 2002.

[11] R.M. Corless, M.W. Giesbrecht, M. van Hoeij, I.S. Kotsireas, and S.M. Watt. Towards factoring bivariate approximate polynomials. In B. Mourrain, editor, *Proceedings of the 2001 International Symposium on Symbolic and Algebraic Computation (ISSAC 2001)*, pages 85–92. ACM, 2001.

[12] D. Cox. What is a toric variety? In R. Goldman and R. Krasauskas, editors, *Topics in Algebraic Geometry and Geometric Modeling*, volume 334 of *Contemporary Mathematics*, pages 203–223. AMS, 2003.

[13] A. Cuyt and W.-s. Lee. A new algorithm for sparse interpolation of multivariate polynomials. *Theoretical Computer Science*, 409(2):180–185, 2008.

[14] T. de Jong and G. Pfister. *Local Analytic Geometry. Basic Theory and Applications.* Vieweg, 2000.

[15] J. Demmel, I. Dumitriu, and O. Holtz. Toward accurate polynomial evaluation in rounded arithmetic. In L. Pardo, A. Pinkus, E. Süli, and M.J. Todd, editors, *Foundations of Computational Mathematics: Santander 2005*, volume 331 of *London Mathematical Society*, pages 36–105. Cambridge University Press, 2006.

[16] M. Elkadi, A. Galligo, and M. Weimann. Towards toric absolute factorization. *J. Symbolic Computation*, 44(9):1194–1211, 2009. Special issue on Effective Methods in Algebraic Geometry edited by André Galligo, Luis Miguel Pardo and Josef Schicho.

[17] G. Fisher. *Plane Algebraic Curves*, volume 15 of *Student Mathematical Library.* AMS, 2001.

[18] A. Galligo and M. van Hoeij. Approximate bivariate factorization, a geometric viewpoint. In J. Verschelde and S.M. Watt, editors, *SNC'07. Proceedings of the 2007 International Workshop on Symbolic-Numeric Computation*, pages 1–10. ACM, 2007.

[19] S. Gao and A.G.B. Lauder. Decomposition of polytopes and polynomials. *Discrete and Computational Geometry*, 26(1):89–104, 2001.

[20] I.M. Gel'fand, M.M. Kapranov, and A.V. Zelevinsky. *Discriminants, Resultants and Multidimensional Determinants.* Birkhäuser, 1994.

[21] M. Giesbrecht, G. Labahn, and W.-s. Lee. Symbolic-numeric sparse interpolation of multivariate polynomials. In J.-G. Dumas, editor, *Proceedings of the 2006 International Symposium on Symbolic and Algebraic Computation (ISSAC 2006)*, pages 116–123. ACM, 2006.

[22] N.B. Grigg. Factorization of tropical polynomials in one and several variables. Bachelor's degree in mathematics, Department of Mathematics, Brigham Young University, 2007.

[23] R. Hemmecke, E. Hillgarter, and F. Winkler. Casa. In J. Grabmeier, E. Kaltofen, and V. Weispfennig, editors, *Computer Algebra Handbook. Foundations, Applications, Systems*, pages 356–358. Springer-Verlag, 2003.

[24] M.A. Hitz, E. Kaltofen, and Y.N. Lakshman. Efficient algorithms for computing the nearest polynomial with a real root and related problems. In S. Dooley, editor, *Proceedings of the 1999 International Symposium on Symbolic and Algebraic Computation (ISSAC 1999)*, pages 205–212, 1999.

[25] B. Huber and B. Sturmfels. A polyhedral method for solving sparse polynomial systems. *Math. Comp.*, 64(212):1541–1555, 1995.

[26] D. Inaba and T. Sasaki. A numerical study of extended Hensel series. In J. Verschelde and S.M. Watt, editors, *SNC'07. Proceedings of the 2007 International Workshop on Symbolic-Numeric Computation*, pages 103–109. ACM, 2007.

[27] A.N. Jensen. *Algorithmic Aspects of Gröbner Fans and Tropical Varieties.* PhD thesis, Department of Mathematical Sciences, University of Aarhus, 2007.

[28] A.N. Jensen. Computing Gröbner fans and tropical varieties in Gfan. In *Software for Algebraic Geometry*, volume 148 of *The IMA Volumes in Mathematics and Its Applications*, pages 33–46. Springer-Verlag, 2008.

[29] A.N. Jensen, H. Markwig, and T. Markwig. tropical.lib. A SINGULAR 3.0 library for computations in tropical geometry, 2007. The library can be downloaded from `http://www.mathematik.uni-kl.de/~keilen/download/Tropical/tropical.lib`.

[30] A.N. Jensen, H. Markwig, and T. Markwig. An algorithm for lifting points in a tropical variety. *Collectanea Mathematica*, 59(2):129–165, 2008.

[31] G. Jeronimo, G. Matera, P. Solernó, and A. Waissbein. Deformation techniques for sparse systems. *Found. Comput. Math*, 9(1):1–50, 2009.

[32] E. Kaltofen and W.-s. Lee. Early termination in sparse interpolation algorithms. *J. Symbolic Computation*, 36(3–4):365–400, 2003.

[33] E. Kaltofen, J.P. May, Z. Yang, and L. Zhi. Approximate factorization of multivariate polynomials using singular value decomposition. *J. Symbolic Computation*, 43(5):359–376, 2008.

[34] B. Ya. Kazarnovskii. Truncation of systems of polynomial equations, ideals and varieties. *Izvestiya: Mathematics*, 63(3):535–547, 1999.

[35] K. Kendig. *Elementary Algebraic Geometry*, volume 44 of *Graduate Texts in Mathematics.* Springer-Verlag, 1977.

[36] W.-s. Lee. From quotient-difference to generalized eigenvalues and sparse polynomial interpolation. In J. Verschelde and S.M. Watt, editors, *SNC'07. Proceedings of the 2007 International Workshop on Symbolic-Numeric Computation*, pages 110–116. ACM, 2007.

[37] M. Lejeune-Jalabert, B. Teissier, and J.-J. Risler. Clôture intégrale des idéaux et équisingularité. `arXiv:0803.2369v1 [math.CV] 16 Mar 2008`.

[38] M. Leksell and W. Komorowski. Amoeba program: Computing and visualizing amoebas for some complex-valued bivariate expressions. Bachelor's degree in mathematics, Department of Mathematics, Natural- and Computer Science, University-College of Gävle, 2007.

[39] T.Y. Li. Numerical solution of polynomial systems by homotopy continuation methods. In F. Cucker, editor, *Handbook of Numerical Analysis. Volume XI. Special Volume: Foundations of Computational Mathematics*, pages 209–304. North-Holland, 2003.

[40] T.Y. Li and Z. Zeng. A rank-revealing method with updating, downdating and applications. *SIAM J. Matrix Anal. Appl.*, 26(4):918–946, 2005.

[41] G.L. Litvinov. The Maslov dequantization, idempotent and tropical mathematics: a very brief introduction. In G.L. Litvinov and V.P. Maslov, editors, *Idempotent Mathematics and Mathematical Physics*, volume 377 of *Contemporary Mathematics*, pages 1–17. AMS, 2005.

[42] J. Maurer. Puiseux expansion for space curves. *Manuscripta Math.*, 32:91–100, 1980.

[43] J. McDonald. Fractional power series solutions for systems of equations. *Discrete Comput. Geom.*, 27(4):501–529, 2002.

[44] G. Mikhalkin. Real algebraic curves, the moment map and amoebas. *Ann. Math.*, 151(1):309–326, 2000.

[45] G. Mikhalkin. Amoebas of algebraic varieties and tropical geometry. In S. Donaldson, Ya. Eliashberg, and M. Gromov, editors, *Different Faces of Geometry*, volume 3 of *International Mathematical Series*, pages 257–300. Springer-Verlag, 2004.

[46] M. Nisse. Complex tropical localization, coamoebas, and mirror tropical hypersurfaces. `arXiv:0806.1959v1 [math.AG] 11 Jun 2008`.

[47] A. Ostrowski. Über die Bedeutung der Theorie der konvexen Polyeder für die formale Algebra. *Jahresbericht d. Deutschen Math. Ver.*, 30:98–99, 1922. Translated by Michael Abramson in *ACM SIGSAM Bulletin* 33(1):5, 1999.

[48] M. Passare and A. Tsikh. Amoebas: their spines and their contours. In G.L. Litvinov and V.P. Maslov, editors, *Idempotent Mathematics and Mathematical Physics*, volume 377 of *Contemporary Mathematics*, pages 275–288. AMS, 2005.

[49] A. Poteaux. Computing monodromy groups defined by plane curves. In J. Verschelde and S.M. Watt, editors, *SNC'07. Proceedings of the 2007 International Workshop on Symbolic-Numeric Computation*, pages 239–246. ACM, 2007.

[50] A. Poteaux and M. Rybowicz. Towards a symbolic-numeric method to compute Puiseux series: the modular part. `arXiv:0803.3027v1 [cs.SC] 20 Mar 2008`.

[51] A. Poteaux and M. Rybowicz. Good reduction of Puiseux series and complexity of the Newton-Puiseux algorithm over finite fields. In D. Jeffrey, editor, *Proceedings of the 2008 International Symposium on Symbolic and Algebraic Computation (ISSAC 2008)*, pages 239–246. ACM, 2008.

[52] V. Puiseux. Recherches sur les fonctions algébriques. *J. de Math. Pures et Appl.*, 15:365–380, 1850.

[53] G. Regensburger. Max-Plus linear algebra in Maple and generalized solutions for first-order ordinary BVPs via Max-Plus interpolation. In M.M. Maza and S. Watt, editors, *Proceedings of MICA 2008: Milestones in Computer Algebra 2008. A Conference in Honour of Keith Geddes' 60th Birthday*, pages 177–182. ACM, 2008.

[54] J. Richter-Gebert, B. Sturmfels, and T. Theobald. First steps in tropical geometry. In G.L. Litvinov and V.P. Maslov, editors, *Idempotent Mathematics and Mathematical Physics*, volume 377 of *Contemporary Mathematics*, pages 289–317. AMS, 2005.

[55] J.M. Rojas. Why polyhedra matter in non-linear equation solving. In R. Goldman and R. Krasauskas, editors, *Topics in Algebraic Geometry and Geometric Modeling*, volume 334 of *Contemporary Mathematics*, pages 293–320. AMS, 2003.

[56] T. Sasaki. A survey of recent advancements of multivariate Hensel construction and applications. In M.M. Maza and S. Watt, editors, *Proceedings of MICA 2008: Milestones in Computer Algebra 2008. A Conference in Honour of Keith Geddes' 60th Birthday*, pages 113–117. ACM, 2008.

[57] J.R. Sendra, F. Winkler, and S. Pérez-Diaz. *Rational Algebraic Curves. A Computer Algebra Approach*, volume 22 of *Algorithms and Computation in Mathematics*. Springer-Verlag, 2008.

[58] V. Sharma, Z. Du, and C.K. Yap. Robust approximate zeros. In G. Stølting Brodal and S. Leonardi, editors, *Algorithms - ESA 2005, 13th Annual European Symposium, Palma de Mallorca, Spain, October 3-6, 2005, Proceedings*, volume 3669 of *Lecture Notes in Computer Science*, pages 874–886. Springer-Verlag, 2005.

[59] A.J. Sommese, J. Verschelde, and C.W. Wampler. Numerical factorization of multivariate complex polynomials. *Theoretical Computer Science*, 315(2-3):651–669, 2004. Special Issue on Algebraic and Numerical Algorithms edited by I.Z. Emiris, B. Mourrain, and V.Y. Pan.

[60] F. Sottile. Toric ideals, real toric varieties, and the algebraic moment map. In R. Goldman and R. Krasauskas, editors, *Topics in Algebraic Geometry and Geometric Modeling*, volume 334 of *Contemporary Mathematics*, pages 225–240. AMS, 2003. Corrected version of the published article is at `arXiv:math/0212044v3 [math.AG] 18 Apr 2008`.

[61] B. Sturmfels. *Gröbner Bases and Convex Polytopes*, volume 8 of *University Lecture Series*. AMS, 1996.

[62] B. Sturmfels. *Solving Systems of Polynomial Equations*. Number 97 in CBMS Regional Conference Series in Mathematics. AMS, 2002.

[63] B. Sturmfels and J. Yu. Tropical implicitization and mixed fiber polytopes. In *Software for Algebraic Geometry*, volume 148 of *The IMA Volumes in Mathematics and Its Applications*, pages 111–131. Springer-Verlag, 2008.

[64] T. Theobald. Computing amoebas. *Experimental Mathematics*, 11(4):513–526, 2002.

[65] T. Theobald. On the frontiers of polynomial computations in tropical geometry. *J. Symbolic Computation*, 41(12):1360–1375, 2006.

[66] J. Verschelde. Polyhedral methods in numerical algebraic geometry. To appear in Interactions of Classical and Numerical Algebraic Geometry, edited by Dan Bates, GianMario Besana, Sandra Di Rocco, and Charles Wampler, Contemporary Mathematics, 2009, AMS.

[67] J. Verschelde. Algorithm 795: PHCpack: A general-purpose solver for polynomial systems by homotopy continuation. *ACM Trans. Math. Softw.*, 25(2):251–276, 1999. Software available at `http://www.math.uic.edu/~jan`.

[68] J. Verschelde. Toric Newton method for polynomial homotopies. *J. Symbolic Computation*, 29(4–5):777–793, 2000.

[69] J. Verschelde, P. Verlinden, and R. Cools. Homotopies exploiting Newton polytopes for solving sparse polynomial systems. *SIAM J. Numer. Anal.*, 31(3):915–930, 1994.

[70] O. Viro. What is ... an amoeba? *Notices of the AMS*, 49(8):916–917, 2002.

[71] R.J. Walker. *Algebraic Curves*. Princeton University Press, 1950.

[72] P.G. Walsh. A polynomial-time complexity bound for the computation of the singular part of a Puiseux expansion of an algebraic function. *Mathematics of Computation*, 69(231):1167–1182, 2000.

[73] Z. Zeng. Computing multiple roots of inexact polynomials. *Mathematics of Computation*, 74(250):869–903, 2005.

[74] Z. Zeng and B.H. Dayton. The approximate GCD of inexact polynomials. Part II: a multivariate algorithm. In J. Gutierrez, editor, *Proceedings of the 2004 International Symposium on Symbolic and Algebraic Computation (ISSAC 2004)*, pages 320–327, 2004.

[75] G.M. Ziegler. *Lectures on Polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.