Polyhedral Methods for Space Curves Exploiting Symmetry Applied to the Cyclic n-roots Problem*

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Abstract

We present a polyhedral algorithm to manipulate positive dimensional solution sets. Using facet normals to Newton polytopes as pretropisms, we focus on the first two terms of a Puiseux series expansion. The leading powers of the series are computed via the tropical prevariety. This polyhedral algorithm is well suited for exploitation of symmetry, when it arises in systems of polynomials. Initial form systems with pretropisms in the same group orbit are solved only once, allowing for a systematic filtration of redundant data. Computations with cddlib, Gfan, PHCpack, and Sage are illustrated on cyclic n-roots polynomial systems.

Keywords. Algebraic set, Backelin's Lemma, cyclic n-roots, initial form, Newton polytope, polyhedral method, polynomial system, Puiseux series, symmetry, tropism, tropical prevariety.

1 Introduction

We consider a polynomial system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$, $\mathbf{f} = (f_1, f_2, \ldots, f_N)$, $f_i \in \mathbb{C}[\mathbf{x}]$, $i = 1, 2, \ldots, N$. Although in many applications the coefficients of the polynomials are rational numbers, we allow the input system to have approximate complex numbers as coefficients. For $N = n$ (as many equations as unknowns), we expect in general to find only isolated solutions. In this paper we focus on cases $N \geq n$ where the coefficients are so special that $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ has an algebraic set as a solution.

Our approach is based on the following observation: if the solution set of $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ has a space curve, then this space curve extends from $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ to infinity. In particular, the space curve intersects hyperplanes at infinity at isolated points. We start our series development of the

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space curve at these isolated points. Computing series developments for solutions of polynomial systems is a hybrid symbolic-numeric method, appropriate for inputs which consist of approximate numbers (the coefficients) and exact data (the exponents).

In this paper we will make various significant assumptions. First we assume that the algebraic sets we consider are reduced, that is: free of multiplicities. Moreover, an algebraic set of dimension d is in general position with respect to the first d coordinate planes. For example, we assume that a space curve is not contained in a plane perpendicular to the first coordinate axis. Thirdly, we assume the algebraic set of dimension d to intersect the first d coordinate planes at regular solutions.

Our approach consists of two stages. The computation of the candidates for the leading powers of the Puiseux series is followed by the computation of the leading coefficients and the second term of the Puiseux series, if the leading term of the series does not already entirely satisfy the system. Following our assumptions, the second term of the Puiseux series indicates the existence of a space curve. If the system is invariant to permutation of the variables, then it suffices to compute only the generators of the solution orbits. We then develop the Puiseux series only at the generators. Although our approach is directed at general algebraic sets, our approach of exploiting symmetry applies also to the computation of all isolated solutions. Our main example is one family of polynomial systems, the cyclic n-roots system.

Related Work. Our approach is inspired by the constructive proof of the fundamental theorem of tropical algebraic geometry in [32] (an alternative proof is in [39]) and related to finiteness proofs in celestial mechanics [27], [30]. The initial form systems allow the elimination of variables with the application of coordinate transformations, an approach presented in [29] and related to the application of the Smith normal form in [25]. The complexity of polyhedral homotopies is studied in [33] and generalized to affine solutions in [28]. Generalizations of the Newton-Puiseux theorem [43], [58], can be found in [5], [7], [37], [38], [45], and [47]. A symbolic-numeric computation of Puiseux series is described in [40], [41], and [42]. Algebraic approaches to exploit symmetry are [13], [20], [23], and [50]. The cyclic n-roots problem is a benchmark for polynomial system solvers, see e.g: [9], [13], [14], [16], [17], [18], [20], [35], [50], and relevant to operator algebras [10], [26], [54]. Our results on cyclic 12-roots correspond to [46].

Our Contributions. This paper is a thorough revision of the unpublished preprint [2], originating in the dissertation of the first author [1], which extended [3] from the plane to space curves. In [4] we gave a tropical version of Backelin's Lemma in case $n = m^2$, in this paper we generalize to the case $n = \ell m^2$. Our approach improves homotopies to find all isolated solutions. Exploiting symmetry we compute only the generating cyclic n-roots, more efficiently than the symmetric polyhedral homotopies of [57].

2 Initial Forms, Cyclic n-roots, and Backelin's Lemma

In this section we introduce our approach on the cyclic 4-roots problem. For this problem we can compute an explicit representation for the solution curves. This explicit representation as monomials in the independent parameters for positive dimensional solution sets generalizes into the tropical version of Backelin's Lemma.

2.1 Newton Polytopes, Initial Forms, and Tropisms

In this section we first define Newton polytopes, initial forms, pretropisms, and tropisms. The sparse structure of a polynomial system is captured by the sets of exponents and their convex hulls.

Definition 2.1. Formally we denote a polynomial $f \in \mathbb{C}[\mathbf{x}]$ as

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad c_{\mathbf{a}} \in \mathbb{C}^*, \quad \mathbf{x}^{\mathbf{a}} = x_0^{a_0} x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}, \tag{1}$$

and we call the set A of exponents *the support of* f. The convex hull of A is *the Newton polytope of* f. The tuple of supports $\mathbf{A} = (A_1, A_2, \ldots, A_N)$ span the Newton polytopes $\mathbf{P} = (P_1, P_2, \ldots, P_N)$ of the polynomials $\mathbf{f} = (f_1, f_2, \ldots, f_N)$ of the system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$.

The development of a series starts at a solution of an initial form of the system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, with supports that span faces of the Newton polytopes of $\mathbf{f}$.

Definition 2.2. Let $\mathbf{v} \neq \mathbf{0}$, denote $\langle \mathbf{a}, \mathbf{v} \rangle = a_0 v_0 + a_1 v_1 + \cdots + a_{n-1} v_{n-1}$, and let f be a polynomial supported on A. Then, *the initial form of* f *in the direction of* $\mathbf{v}$ is

$$\mathrm{in}_{\mathbf{v}}(f) = \sum_{\mathbf{a} \in \mathrm{in}_{\mathbf{v}}(A)} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad \text{where} \quad \mathrm{in}_{\mathbf{v}}(A) = \{ \, \mathbf{a} \in A \mid \langle \mathbf{a}, \mathbf{v} \rangle = \min_{\mathbf{b} \in A} \langle \mathbf{b}, \mathbf{v} \rangle \, \}. \tag{2}$$

The initial form of a system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ with polynomials in $\mathbf{f} = (f_1, f_2, \ldots, f_N)$ in the direction of $\mathbf{v}$ is denoted by $\mathrm{in}_{\mathbf{v}}(\mathbf{f}) = (\mathrm{in}_{\mathbf{v}}(f_1), \mathrm{in}_{\mathbf{v}}(f_2), \ldots, \mathrm{in}_{\mathbf{v}}(f_N))$. If the number of monomials with nonzero coefficient in each $\mathrm{in}_{\mathbf{v}}(f_k)$, for all $k = 1, 2, \ldots, N$, is at least two, then $\mathbf{v}$ is a *pretropism*.

The notation $\mathrm{in}_{\mathbf{v}}(f)$ follows [53], where $\mathbf{v}$ represents a weight vector to order monomials. The polynomial $\mathrm{in}_{\mathbf{v}}(f)$ is homogeneous with respect to $\mathbf{v}$. Therefore, in solutions of $\mathrm{in}_{\mathbf{v}}(f)(\mathbf{x}) = \mathbf{0}$ we can set x_0 to the free parameter t. In [12] and [34], initial form systems are called truncated systems.

Faces of Newton polytopes P spanned by two points are edges and all vectors $\mathbf{v}$ that lead to the same $\mathrm{in}_{\mathbf{v}}(P)$ (the convex hull of $\mathrm{in}_{\mathbf{v}}(A)$) define a polyhedral cone (see e.g. [59] for an introduction to polytopes).

Definition 2.3. Given a tuple of Newton polytopes $\mathbf{P}$ of a system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, *the tropical prevariety* of $\mathbf{f}$ is the common refinement of the normal cones to the edges of the Newton polytopes in $\mathbf{P}$.

Our definition of a tropical prevariety is based on the algorithmic characterization in [11, Algorithm 2], originating in [44]. Consider for example the special case of two polytopes P_1 and P_2 and take the intersection of two cones, normal to two edges of the two polytopes. If the intersection is not empty, then the intersection contains a vector $\mathbf{v}$ that defines a tuple of two edges $(\mathrm{in}_{\mathbf{v}}(P_1), \mathrm{in}_{\mathbf{v}}(P_2))$.

Definition 2.4. For space curves, the special role of x_0 is reflected in the normal form of the Puiseux series:

$$\left\{ \begin{array}{l} x_0 = t^{v_0} \\ x_i = t^{v_i}(y_i + z_i t^{w_i}(1 + O(t)), \quad i = 1, 2, \ldots, n-1, \end{array} \right. \tag{3}$$

where the leading powers $\mathbf{v} = (v_0, v_1, \ldots, v_{n-1})$ define a *tropism*.

In the definition above, it is important to observe that the tropism $\mathbf{v}$ defines as well the initial form system $\text{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{x}) = \mathbf{0}$ that has as solution the initial coefficients of the Puiseux series.

Every tropism is a pretropism, but not every pretropism is a tropism, because pretropisms depend only on the Newton polytopes. For a d-dimensional algebraic set, a d-dimensional polyhedral cone of tropisms defines the exponents of Puiseux series depending on d free parameters.

2.2 The Cyclic n-roots Problem

For $n = 3$, the cyclic n-roots system originates naturally from the elementary symmetric functions in the roots of a cubic polynomial. For $n = 4$, the system is

$$\mathbf{f}(\mathbf{x}) = \begin{cases} x_0 + x_1 + x_2 + x_3 = 0 \\ x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 = 0 \\ x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_3x_0x_1 = 0 \\ x_0x_1x_2x_3 - 1 = 0. \end{cases} \tag{4}$$

The permutation group which leaves the equations invariant is generated by $(x_0, x_1, x_2, x_3) \to (x_1, x_2, x_3, x_0)$ and $(x_0, x_1, x_2, x_3) \to (x_3, x_2, x_1, x_0)$. In addition, the system is equi-invariant with respect to the action $(x_0, x_1, x_2, x_3) \to (x_0^{-1}, x_1^{-1}, x_2^{-1}, x_3^{-1})$.

With $\mathbf{v} = (+1, -1, +1, -1)$, there is a unimodular coordinate transformation M, denoted by $\mathbf{x} = \mathbf{z}^M$:

$$\text{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{x}) = \begin{cases} x_1 + x_3 = 0 \\ x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 = 0 \\ x_1x_2x_3 + x_3x_0x_1 = 0 \\ x_0x_1x_2x_3 - 1 = 0 \end{cases} \qquad \mathbf{x} = \mathbf{z}^M : \begin{cases} x_0 = z_0^{+1} \\ x_1 = z_0^{-1}z_1 \\ x_2 = z_0^{+1}z_2 \\ x_3 = z_0^{-1}z_3 \end{cases} \tag{5}$$

with

$$M = \begin{bmatrix} +1 & -1 & +1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6}$$

The system $\text{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{z}) = \mathbf{0}$ has two solutions. These two solutions are the leading coefficients in the Puiseux series. In this case, the leading term of the series vanishes entirely at the system so we write two solution curves as $\left(t, -t^{-1}, -t, t^{-1}\right)$ and $\left(t, t^{-1}, -t, -t^{-1}\right)$. To compute the degree of the two solution curves, we take a random hyperplane in $\mathbb{C}^4$: $c_0x_0 + c_1x_1 + c_2x_2 + c_3x_3 + c_5 = 0$, $c_i \in \mathbb{C}^*$. Then the number of points on the curve and on the random hyperplane equals the degree of the curve. Substituting the representations we obtained for the curves into the random hyperplanes gives a quadratic polynomial in t (after clearing the denominator t^{-1}), so there are two quadric curves of cyclic 4-roots.

2.3 A Tropical Version of Backelin's Lemma

In [4], we gave an explicit representation for the solution sets of cyclic n-roots, in case $n = m^2$, for any natural number $m \geq 2$. Below we state Backelin's Lemma [6], in its tropical form.

Lemma 2.5 (Tropical Version of Backelin's Lemma). *For $n = m^2\ell$, where $\ell \in \mathbb{N} \setminus \{0\}$ and ℓ is no multiple of k^2, for $k \geq 2$, there is an $(m-1)$-dimensional set of cyclic n-roots, represented exactly as*

$$\begin{array}{rcl} x_{km+0} & = & u^k t_0 \\ x_{km+1} & = & u^k t_0 t_1 \\ x_{km+2} & = & u^k t_0 t_1 t_2 \\ & \vdots & \\ x_{km+m-2} & = & u^k t_0 t_1 t_2 \cdots t_{m-2} \\ x_{km+m-1} & = & \gamma u^k t_0^{-m+1} t_1^{-m+2} \cdots t_{m-3}^{-2} t_{m-2}^{-1} \end{array} \tag{7}$$

for $k = 0, 1, 2, \ldots, m-1$, free parameters $t_0, t_1, \ldots, t_{m-2}$, constants $u = e^{\frac{i2\pi}{m\ell}}$, $\gamma = e^{\frac{i\pi\beta}{m\ell}}$, with $\beta = (\alpha \mod 2)$, and $\alpha = m(m\ell - 1)$.

Proof. By performing the change of variables $y_0 = t_0$, $y_1 = t_0 t_1$, $y_2 = t_0 t_1 t_2$, ..., $y_{m-2} = t_0 t_1 t_2 \cdots t_{m-2}$, $y_{m-1} = \gamma t_0^{-m+1} t_1^{-m+2} \cdots t_{m-3}^{-2} t_{m-2}^{-1}$, the solution (7) can be rewritten as

$$x_{km+j} = u^k y_j, \quad j = 0, 1, \ldots, m-1. \tag{8}$$

The solution (8) satisfies the cyclic n-roots system by plain substitution as in the proof of [19, Lemma 1.1], whenever the last equation $x_0 x_1 x_2 \cdots x_{n-1} - 1 = 0$ of the cyclic n-roots problem can also be satisfied.

We next show that we can always satisfy the equation $x_0 x_1 x_2 \cdots x_{n-1} - 1 = 0$ with our solution. First, we perform an additional change of coordinates to separate the γ coefficient. We let $y_0 = Y_0$, $y_1 = Y_1$, ..., $y_{m-2} = Y_{m-2}$, $y_{m-1} = \gamma Y_{m-1}$. Then on substitution of (8) into $x_0 x_1 x_2 \cdots x_{n-1} - 1 = 0$, we get

$$\begin{array}{rcl} (\gamma^{m\ell} \ \underbrace{u^0 u^0 \cdots u^0}_{m} \underbrace{u^1 u^1 \cdots u^1}_{m} \cdots \underbrace{u^{m\ell-1} u^{m\ell-1} \cdots u^{m\ell-1}}_{m} \ Y_0^{m\ell} Y_1^{m\ell} Y_2^{m\ell} \cdots Y_{m-2}^{m\ell} Y_{m-1}^{m\ell}) - 1 & = & 0 \\ (\gamma^{m\ell} \ u^{m(0+1+2+\cdots+(m\ell-1))} \ Y_0^{m\ell} Y_1^{m\ell} Y_2^{m\ell} \cdots Y_{m-2}^{m\ell} Y_{m-1}^{m\ell}) - 1 & = & 0 \\ (\gamma \ u^{\frac{m(m\ell-1)}{2}} \ Y_0 Y_1 Y_2 \cdots Y_{m-2} Y_{m-1})^{m\ell} - 1 & = & 0. \end{array} \tag{9}$$

The last equation in (9) has now the same form as in [19, Lemma 1.1]. We are done if we can satisfy it. We next show that it can always be satisfied with our solution.

Since all the tropisms in the cone add up to zero, the product $(Y_0 Y_1 Y_2 \cdots Y_{m-2} Y_{m-1})$, which consists of free parameter combinations, equals to 1. Since $(Y_0 Y_1 Y_2 \cdots Y_{m-2} Y_{m-1}) = 1$, we are left with

$$(\gamma \ u^{\frac{m(m\ell-1)}{2}})^{m\ell} - 1 = 0. \tag{10}$$

We distinguish two cases:

1. $\gamma = 1$, implied by (m is even, ℓ is odd) or (m is odd, ℓ is odd) or (m is even, ℓ is even).

To show that (10) is satisfied, we rewrite (10):

$$(u^{\frac{m(m\ell-1)}{2}})^{m\ell} - 1 = 0 \quad \Leftrightarrow \quad (u^{\frac{m^2\ell(m\ell-1)}{2}}) - 1 = 0 \quad \Leftrightarrow \quad ((u^{m\ell})^{\frac{m(m\ell-1)}{2}}) - 1 = 0, \tag{11}$$

which is satisfied by $u = e^{\frac{i2\pi}{m\ell}}$ and $m(m\ell - 1)$ being even.

2. $\gamma \neq 1$, implied by (m is odd, ℓ is even).

   To show that our solution satisfies (10), we rewrite (10):

$$(\gamma\ u^{\frac{m(m\ell-1)}{2}})^{m\ell} - 1 = 0 \quad \Leftrightarrow \quad (\gamma\ u^{\frac{m^2\ell}{2}}\ u^{\frac{-m}{2}})^{m\ell} - 1 = 0 \quad \Leftrightarrow \quad (\gamma\ (u^{m\ell})^{\frac{m}{2}}\ u^{\frac{-m}{2}})^{m\ell} - 1 = 0. \tag{12}$$

   Since $u = e^{\frac{i2\pi}{m\ell}}$, $u^{m\ell} = 1$, we can simplify (12) further

$$\begin{array}{rcl} (\gamma\ u^{\frac{-m}{2}})^{m\ell} - 1 & = & 0 \\ (e^{\frac{i\pi}{m\ell}}\ (e^{\frac{i2\pi}{m\ell}})^{\frac{-m}{2}})^{m\ell} - 1 & = & 0 \\ (e^{\frac{i\pi}{m\ell}}\ (e^{\frac{-i\pi}{\ell}}))^{m\ell} - 1 & = & 0 \\ (e^{i\pi}\ e^{-i\pi m}) - 1 & = & 0 \\ (e^{(1-m)i\pi}) - 1 & = & 0. \end{array} \tag{13}$$

   Since m is odd, we can write $m = 2j+1$, for some j. The last equation of (13) has the form

$$(e^{(1-m)i\pi}) - 1 = 0 \quad \Leftrightarrow \quad (e^{(1-(2j+1))i\pi}) - 1 = 0 \quad \Leftrightarrow \quad (e^{(-2j)i\pi}) - 1 = 0. \tag{14}$$

   Since $(e^{(-2j)i\pi}) = 1$, for any j, the equation $(e^{(-2j)i\pi}) - 1 = 0$ is satisfied, implying (10).

□

Backelin's Lemma comes to aid when applying a homotopy to find all isolated cyclic n-roots as follows. We must decide at the end of a solution path whether we have reached an isolated solution or a positive dimension solution set. This problem is especially difficult in the presence of isolated singular solutions (such as 4-fold isolated cyclic 9-roots [36]). With the form of the solution set as in Backelin's Lemma, we solve a triangular binomial system in the parameters t and with as x values the solution found at the end of a path. If we find values for the parameters for an end point, then this solution lies on the solution set.

3 Exploiting Symmetry

We illustrate the exploitation of permutation symmetry on the cyclic 5-roots system. Adjusting polyhedral homotopies to exploit the permutation symmetry for this system was presented in [57].

3.1 The Cyclic 5-roots Problem

The mixed volume for the cyclic 5-roots system is 70, which equals the exact number of roots. The first four equations of the cyclic 5-roots system $C_5(\mathbf{x}) = \mathbf{0}$, define solution curves:

$$\mathbf{f}(\mathbf{x}) = \begin{cases} x_0 + x_1 + x_2 + x_3 + x_4 = 0 \\ x_0x_1 + x_0x_4 + x_1x_2 + x_2x_3 + x_3x_4 = 0 \\ x_0x_1x_2 + x_0x_1x_4 + x_0x_3x_4 + x_1x_2x_3 + x_2x_3x_4 = 0 \\ x_0x_1x_2x_3 + x_0x_1x_2x_4 + x_0x_1x_3x_4 + x_0x_2x_3x_4 + x_1x_2x_3x_4 = 0. \end{cases} \tag{15}$$

where $\mathbf{v} = (1,1,1,1,1)$. As the first four equations of C_5 are homogeneous, the first four equations of C_5 coincide with the first four equations of $\text{in}_{\mathbf{v}}(C_5)(\mathbf{x}) = \mathbf{0}$. Because these four equations are homogeneous, we have lines of solutions. After computing representations for the solution lines, we find the solutions to the original cyclic 5-roots problem intersecting the solution lines with the hypersurface defined by the last equation. In this intersection, the exploitation of the symmetry is straightforward.

The unimodular matrix with $\mathbf{v} = (1,1,1,1,1)$ and its corresponding coordinate transformation are

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{x} = \mathbf{z}^M : \begin{cases} x_0 = z_0 \\ x_1 = z_0z_1 \\ x_2 = z_0z_2 \\ x_3 = z_0z_3 \\ x_4 = z_0z_4. \end{cases} \tag{16}$$

Applying $\mathbf{x} = \mathbf{z}^M$ to the initial form system (15) gives

$$\text{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{x} = \mathbf{z}^M) = \begin{cases} z_1 + z_2 + z_3 + z_4 + 1 = 0 \\ z_1z_2 + z_2z_3 + z_3z_4 + z_1 + z_4 = 0 \\ z_1z_2z_3 + z_2z_3z_4 + z_1z_2 + z_1z_4 + z_3z_4 = 0 \\ z_1z_2z_3z_4 + z_1z_2z_3 + z_1z_2z_4 + z_1z_3z_4 + z_2z_3z_4 = 0. \end{cases} \tag{17}$$

The system (17) has 14 isolated solutions of the form $z_1 = c_1,\ \ z_2 = c_2,\ \ z_3 = c_3,\ \ z_4 = c_4$. If we let $z_0 = t$, in the original coordinates we have

$$x_0 = t,\ x_1 = tc_1,\ x_2 = tc_2,\ x_3 = tc_3,\ x_4 = tc_4 \tag{18}$$

as representations for the 14 solution lines.

Substituting (18) into the omitted equation $x_0x_1x_2x_3x_4 - 1 = 0$, yields a univariate polynomial in t of the form $kt^5 - 1 = 0$, where k is a constant. Among the 14 solutions, 10 are of the form $t^5 - 1$. They account for $10 \times 5 = 50$ solutions. There are two solutions of the form $(-122.99186938124345)t^5 - 1$, accounting for $2 \times 5 = 10$ solutions and an additional two solutions are of the form $(-0.0081306187557833118)t^5 - 1$ accounting for $2 \times 5 = 10$ remaining solutions. The total number of solutions is 70, as indicated by the mixed volume computation. Existence of additional symmetry, which can be exploited, can be seen in the relationship between the coefficients of the quintic polynomial, i.e. $\frac{1}{(-122.99186938124345)} \approx -0.0081306187557833118$.

3.2 A General Approach

That the first $n-1$ equations of cyclic n-roots system give explicit solution lines is exceptional. For general polynomial systems we can use the leading term of the Puiseux series to compute witness sets [49] for the space curves defined by the first $n-1$ equations. Then via the diagonal homotopy [48] we can intersect the space curves with the rest of the system. While the direct exploitation of symmetry with witness sets is not possible, with the Puiseux series we can pick out the generating space curves.

4 Computing Pretropisms

Following from the second theorem of Bernshteĭn [8], the Newton polytopes may be in general position and no normals to at least one edge of every Newton polytope exists. In that case, there does not exist a positive dimensional solution set either. We look for $\mathbf{v}$ so that $\text{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{x}) = \mathbf{0}$ has solutions in $(\mathbb{C}^*)^n$ and therefore we look for pretropisms. In this section we describe two approaches to compute pretropisms. The first approach applies cddlib [21] on the Cayley embedding. Algorithms to compute tropical varieties are described in [11] and implemented in Gfan [31]. The second approach is the application of `tropical_intersection` of Gfan.

4.1 Using the Cayley Embedding

The Cayley trick formulates a resultant as a discriminant as in [24, Proposition 1.7, page 274]. We follow the geometric description of [52], see also [15, §9.2]. The *Cayley embedding* $E_{\mathbf{A}}$ *of* $\mathbf{A} = (A_1, A_2, \ldots, A_N)$ is

$$E_{\mathbf{A}} = (A_1 \times \{\mathbf{0}\}) \cup (A_2 \times \{\mathbf{e}_1\}) \cup \cdots \cup (A_N \times \{\mathbf{e}_{N-1}\}) \tag{19}$$

where $\mathbf{e}_k$ is the kth $(N-1)$-dimensional unit vector. Consider the convex hull of the Cayley embedding, the so-called Cayley polytope, denoted by $\text{conv}(E_{\mathbf{A}})$. If $\dim(E_{\mathbf{A}}) = k < 2n-1$, then a facet of $\text{conv}(E_{\mathbf{A}})$ is a face of dimension $k-1$.

Proposition 4.1. *Let $E_{\mathbf{A}}$ be the Cayley embedding of the supports $\mathbf{A}$ of the system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. The normals of those facets of $\text{conv}(E_{\mathbf{A}})$ that are spanned by at least two points of each support in $\mathbf{A}$ form the tropical prevariety of $\mathbf{f}$.*

Proof. Denote the Minkowski sum of the supports in $\mathbf{A}$ as $\Sigma_{\mathbf{A}} = A_1 + A_2 + \cdots + A_N$. Facets of $\Sigma_{\mathbf{A}}$ spanned by at least two points of each support define the generators of the cones of the tropical prevariety. The relation between $E_{\mathbf{A}}$ and $\Sigma_{\mathbf{A}}$ is stated explicitly in [15, Observation 9.2.2]. In particular, cells in a polyhedral subdivision of $E_{\mathbf{A}}$ are in one-to-one correspondence with cells in a polyhedral subdivision of the Minkowski sum $\Sigma_{\mathbf{A}}$. The correspondence with cells in a polyhedral subdivision implies that facet normals of $\Sigma_{\mathbf{A}}$ occur as facet normals of $\text{conv}(E_{\mathbf{A}})$. Thus the set of all facets of $\text{conv}(E_{\mathbf{A}})$ gives the tropical prevariety of $\mathbf{f}$. □

Note that $\Sigma_{\mathbf{A}}$ can be computed as the Newton polytope of the product of all polynomials in $\mathbf{f}$. As a practical matter, applying the Cayley embedding is better than just plainly computing the convex hull of the Minkowski sum because the Cayley embedding maintains the sparsity of the input, at the expense of increasing the dimension. Running `cddlib` [21] to compute the H-representation of the Cayley polytope of the cyclic 8-roots problem yields 94 pretropisms. With symmetry we have 11 generators, displayed in Table 1.

generating pretropisms and initial forms			higher dimensional cones of pretropisms			
	pretropism $\mathbf{v}$	#solutions of $\text{in}_{\mathbf{v}}(C_8)(\mathbf{z})$	1D	2D	3D	4D
1.	$(-3,1,1,1,-3,1,1,1)$	94	{1}	{1, 3}	{1, 6, 11}	{1, 2, 3, 11}
2.	$(-1,-1,-1,3,-1,-1,-1,3)$	115	{2}	{1, 6}	{1, 10, 11}	
3.	$(-1,-1,1,1,-1,-1,1,1)$	112	{3}	{1, 10}	{2, 8, 11}	
4.	$(-1,0,0,0,1,-1,1,0)$	30	{4}	{1, 11}		
5.	$(-1,0,0,0,1,0,-1,1)$	23	{5}	{2, 3}		
6.	$(-1,0,0,1,-1,1,0,0)$	32	{6}	{2, 8}		
7.	$(-1,0,0,1,0,-1,1,0)$	40	{7}	{3, 7}		
8.	$(-1,0,0,1,0,0,-1,1)$	16	{8}	{2, 11}		
9.	$(-1,0,1,-1,1,-1,1,0)$	39	{9}	{6, 11}		
10.	$(-1,0,1,0,-1,1,-1,1)$	23	{10}	{8, 11}		
11.	$(-1,1,-1,1,-1,1,-1,1)$	509	{11}	{10, 11}		

Table 1: Eleven pretropism generators of the cyclic 8-root problem, the number of solutions of the corresponding initial form systems, and the multidimensional cones they generate, as computed by Gfan.

For the cyclic 9-roots problem, the computation of the facets of the Cayley polytope yield 276 pretropisms, with 17 generators: $(-2, 1, 1, -2, 1, 1, -2, 1, 1)$, $(-1, -1, 2, -1, -1, 2, -1, -1, 2)$, $(-1, 0, 0, 0, 0, 1, -1, 1, 0)$, $(-1, 0, 0, 0, 0, 1, 0, -1, 1)$, $(-1, 0, 0, 0, 1, -1, 0, 1, 0)$, $(-1, 0, 0, 0, 1, -1, 1, 0, 0)$, $(-1, 0, 0, 0, 1, 0, -1, 0, 1)$, $(-1, 0, 0, 0, 1, 0, -1, 1, 0)$, $(-1, 0, 0, 0, 1, 0, 0, -1, 1)$, $(-1, 0, 0, 1, -1, 0, 1, -1, 1)$, $(-1, 0, 0, 1, -1, 0, 1, 0, 0)$, $(-1, 0, 0, 1, -1, 1, -1, 0, 1)$, $(-1, 0, 0, 1, -1, 1, -1, 1, 0)$, $(-1, 0, 0, 1, -1, 1, 0, -1, 1)$, $(-1, 0, 0, 1, 0, -1, 1, -1, 1)$, $(-1, 0, 0, 1, 0, 0, -1, 0, 1)$, and $(-1, 0, 1, -1, 1, -1, 0, 1, 0)$. To get the structure of the two dimensional cones, a second run of the Cayley embedding is needed on the smaller initial form systems defined by the pretropisms.

The computations for $n = 8$ and $n = 9$ finished in less than a second on one core of a 3.07Ghz Linux computer with 4Gb RAM. For the cyclic 12-roots problem, `cddlib` needed about a week to compute the 907,923 facets normals of the Cayley polytope. Although effective, the Cayley embedding becomes too inefficient for larger problems.

4.2 Using `tropical_intersection` of Gfan

The solution set of the cyclic 8-roots polynomial system consists of space curves. Therefore, all tropisms cones were generated by a single tropism. The computation of the tropical prevariety however, did not lead only to single pretropisms but also to cones of pretropisms. The cyclic 8-roots cones of pretropisms and their dimension are listed in Table 1. Since the one dimensional rays of pretropisms yielded initial form systems with isolated solutions and since all higher dimensional cones are spanned by those one dimensional rays, we can conclude that there are no higher dimensional algebraic sets, as any two dimensional surface degenerates to a curve if we consider only one tropism.

For the computation of the tropical prevariety, the Sage 5.7/Gfan function `tropical_intersection()` ran (with default settings without exploitation of symmetry) on an AMD Phenom II X4 820 processor with 6 GB of RAM, running GNU/Linux, see Table 2. As the dimension n increases so does the running time, but the relative cost factors are bounded by n.

n	seconds	hms format	factor
8	16.37	16 s	1.0
9	79.36	1 m 19 s	4.8
10	503.53	8 m 23 s	6.3
11	3898.49	1 h 4 m 58 s	7.7
12	37490.93	10 h 24 m 50 s	9.6

Table 2: Time to compute the tropical prevarieties for cyclic n-roots with Sage 5.7/Gfan and the relative cost factors: for $n = 12$, it takes 9.6 times longer than for $n = 11$.

5 The Second Term of a Puiseux Series

In exceptional cases like the cyclic 4-roots problem where the first term of the series gives an exact solution or when we encounter solution lines like with the first four equations of cyclic 5-roots, we do not have to look for a second term of a series. In general, a pretropism $\mathbf{v}$ becomes a tropism if there is a Puiseux series with leading powers equal to $\mathbf{v}$. The leading coefficients of the series is a solution in $\mathbb{C}^*$ of the initial form system $\text{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{x}) = \mathbf{0}$. We solve the initial form systems with PHCpack [55] (its blackbox solver incorporates MixedVol [22]). For the computations of the series we use Sage [51].

5.1 Computing the Second Term

In our approach, the calculation of the second term in the Puiseux series is critical to decide whether a solution of an initial form system corresponds to an isolated solution at infinity of the original system, or whether it constitutes the beginning of a space curve. For sparse systems, we may not assume that the second term of the series is linear in t. Trying consecutive powers of t will be wasteful for high degree second terms of particular systems. In this section we explain our algorithm to compute the second term in the Puiseux series.

A unimodular coordinate transformation $\mathbf{x} = \mathbf{z}^M$ with M having as first row the vector $\mathbf{v}$ turns the initial form system $\text{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{x}) = \mathbf{0}$ into $\text{in}_{\mathbf{e}_1}(\mathbf{f})(\mathbf{z}) = \mathbf{0}$ where $\mathbf{e}_1 = (1, 0, \ldots, 0)$ equals the first standard basis vector. When $\mathbf{v}$ has negative components, solutions of $\text{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{x}) = \mathbf{0}$ that are at infinity (in the ordinary sense of having components equal to ∞) are turned into solutions in $(\mathbb{C}^*)^n$ of $\text{in}_{\mathbf{e}_1}(\mathbf{f})(\mathbf{z}) = \mathbf{0}$.

The following proposition states the existence of the exponent of the second term in the series. After the proof of the proposition we describe how to compute this second term.

Proposition 5.1. *Let* $\mathbf{v}$ *denote the pretropism and* $\mathbf{x} = \mathbf{z}^M$ *denote the unimodular coordinate transformation, generated by* $\mathbf{v}$*. Let* $\mathrm{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{x} = \mathbf{z}^M)$ *denote the transformed initial form system with regular isolated solutions, forming the isolated solutions at infinity of the transformed polynomial system* $\mathbf{f}(\mathbf{x} = \mathbf{z}^M)$*. If the substitution of the regular isolated solutions at infinity into the transformed polynomial system* $\mathbf{f}(\mathbf{x} = \mathbf{z}^M)$ *does not satisfy the system entirely, then the constant terms of* $\mathbf{f}(\mathbf{x} = \mathbf{z}^M)$ *have disappeared, leaving at least one monomial* $c_\ell t^{w_\ell}$ *for some* f_ℓ *in* $\mathbf{f}(\mathbf{x} = \mathbf{z}^M)$ *with minimal value* w_ℓ*. The minimal exponent* w_ℓ *is the candidate for the exponent of the second term in the Puiseux series.*

Proof. Let $\mathbf{z} = (z_0, z_1, \ldots, z_{n-1})$ and $\bar{\mathbf{z}} = (z_1, z_2, \ldots, z_{n-1})$ denote variables after the unimodular transformation. Let $(z_0 = t, z_1 = r_1, \ldots, z_{n-1} = r_{n-1})$ be a regular solution at infinity and t the free variable.

The ith equation of the original system after the unimodular coordinate transformation has the form

$$f_i = z_0^{m_i}(P_i(\bar{\mathbf{z}}) + O(z_0)Q_i(\mathbf{z})), \quad i = 1, 2, \ldots, N, \tag{20}$$

where the polynomial $P_i(\bar{\mathbf{z}})$ consists of all monomials which form the initial form component of f_i and $Q_i(\mathbf{z})$ is a polynomial consisting of all remaining monomials of f_i. After the coordinate transformation, we denote the series expansion as

$$\left\{ \begin{array}{l} z_0 = t \\ z_j = r_j + k_j t^{w_\ell}(1 + O(t)), \quad j = 1, 2, \ldots, n-1. \end{array} \right. \tag{21}$$

for some ℓ and where at least one k_j is nonzero.

We first show that, for all i, the polynomial $z_0^{m_i} P_i(\bar{\mathbf{z}})$ cannot contain a monomial of the form $c_\ell t^{w_\ell}$ on substitution of (21). The polynomial $z_0^{m_i} P_i(\bar{\mathbf{z}})$ is the initial form of f_i, hence solution at infinity $(z_0 = t, z_1 = r_1, z_2 = r_2, \ldots, z_{n-1} = r_{n-1})$ satisfies $z_0^{m_i} P_i(\bar{\mathbf{z}})$ entirely. Substituting (21) into $z_0^{m_i} P_i(\bar{\mathbf{z}})$ eliminates all constants in $t^{m_i} P_i(\bar{\mathbf{z}})$. Hence, the polynomial $P_i(t) = R_i(t^w)$ and, therefore, $t^{m_i} P_i(t) = R_i(t^{w+m_i})$.

We next show that for some $i = \ell$, the polynomial $Q_i(\mathbf{z})$ contains a monomial $c_\ell t^{w_\ell}$. The polynomial $Q_i(\mathbf{z})$ is rewritten:

$$z_0^{w_\ell} Q_i(\bar{\mathbf{z}}) = z_0^{w_\ell} T_{i0}(\bar{\mathbf{z}}) + z_0^{w_\ell + 1} T_{i1}(\bar{\mathbf{z}}) + \cdots. \tag{22}$$

The polynomial $Q_i(\mathbf{z}) = z_0^{w_\ell} Q_i(\bar{\mathbf{z}})$ consists of monomials which are not part of the initial form of f_i. Hence, on substitution of solution at infinity (21), $z_0^{w_\ell} Q_i(\bar{\mathbf{z}}) = t^{w_\ell} Q_i(t)$ does not vanish entirely and there must be at least one $i = \ell$ for which constants remain after substitution. Since $Q_\ell(t)$ contains monomials which are constants, $t^{w_\ell} Q_\ell(t)$ must contain a monomial of the form $c_\ell t^{w_\ell}$. □

Now we describe the computation of the second term, in case the initial root does not satisfy the entire original system. Assume the following general form of the series:

$$\left\{ \begin{array}{l} z_0 = t \\ z_i = c_i^{(0)} + k_i t^{w_\ell}(1 + O(t)), \quad i = 1, 2, \ldots, n-1, \end{array} \right. \tag{23}$$

for some ℓ and where $c_i^{(0)} \in \mathbb{C}^*$ are the coordinates of the initial root, k_i is the unknown coefficient of the second term t^{w_ℓ}, $w_\ell > 0$. Note that only for some k_i nonzero values may exist, but not all k_i may be zero. We are looking for the smallest w_ℓ for which the linear system in the k_i's admits a solution with at least one nonzero coordinate. Substituting (23) gives equations of the form

$$\widehat{c_i^{(0)}} t^{w_\ell}(1 + O(t)) + t^{w_\ell + b_i} \sum_{j=1}^{n} \gamma_{ij} k_j (1 + O(t)) = 0, \quad i = 1, 2, \ldots, n, \tag{24}$$

for constant exponents w_ℓ, b_i and constant coefficients $\widehat{c_i^{(0)}}$ and γ_{ij}.

In the equations of (24) we truncate the $O(t)$ terms and retain those equations with the smallest value of the exponents w_ℓ, because with the second term of the series solution we want to eliminate the lowest powers of t when we plug in the first two terms of the series into the system. This gives a condition on the value w_ℓ of the unknown exponent of t in the second term. If there is no value for w_ℓ so that we can match with $w_\ell + b_i$ the minimal value of w_ℓ for all equations where the same minimal value of w_ℓ occurs, then there does not exist a second term and hence no space curve. Otherwise, with the matching value for w_ℓ we obtain a linear system in the unknown k variables. If a solution to this linear system exists with at least one nonzero coordinate, then we have found a second term, otherwise, there is no space curve.

For an algebraic set of dimension d, we have a polyhedral cone of d tropisms and we take any general vector $\mathbf{v}$ in this cone. Then we apply the method outlined above to compute the second term in the series in one parameter, in the direction of $\mathbf{v}$.

5.2 Series Developments for Cyclic 8-roots

We illustrate our approach on the cyclic 8-roots problem, denoted by $C_8(\mathbf{x}) = \mathbf{0}$ and take as pretropism $\mathbf{v} = (1, -1, 0, 1, 0, 0, -1, 0)$. Replacing the first row of the 8-dimensional identity matrix by $\mathbf{v}$ yields a unimodular coordinate transformation, denoted as $\mathbf{x} = \mathbf{z}^M$, explicitly defined as

$$x_0 = z_0,\ x_1 = z_1/z_0,\ x_2 = z_2,\ x_3 = z_0 z_3,\ x_4 = z_4,\ x_5 = z_5,\ x_6 = z_6/z_0,\ x_7 = z_7. \tag{25}$$

Applying $\mathbf{x} = \mathbf{z}^M$ to the initial form system $\mathrm{in}_{\mathbf{v}}(C_8)(\mathbf{x}) = \mathbf{0}$ gives

$$\mathrm{in}_{\mathbf{v}}(C_8)(\mathbf{x} = \mathbf{z}^M) = \begin{cases} z_1 + z_6 = 0 \\ z_1 z_2 + z_5 z_6 + z_6 z_7 = 0 \\ z_4 z_5 z_6 + z_5 z_6 z_7 = 0 \\ z_4 z_5 z_6 z_7 + z_1 z_6 z_7 = 0 \\ z_1 z_2 z_6 z_7 + z_1 z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 z_6 + z_1 z_2 z_5 z_6 z_7 + z_1 z_4 z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 z_6 z_7 + z_1 z_2 z_4 z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 z_6 z_7 - 1 = 0. \end{cases} \tag{26}$$

By construction of M, observe that all polynomials have the same power of z_0, so z_0 can be factored out. Removing z_0 from the initial form system, we find a solution

$$z_0 = t,\ z_1 = -I,\ z_2 = \frac{-1}{2} - \frac{I}{2},\ z_3 = -1,\ z_4 = 1 + I,\ z_5 = \frac{1}{2} + \frac{I}{2},\ z_6 = I,\ z_7 = -1 - I \tag{27}$$

where $I = \sqrt{-1}$. This solution is a regular solution. We set $z_0 = t$, where t is the variable for the Puiseux series. In the computation of the second term, we assume the Puiseux series of the form at the left of (28). We first transform the cyclic 8-roots system $C_8(\mathbf{x}) = \mathbf{0}$ using the coordinate transformation given by (25) and then substitute the assumed series form into this new system. Since the next term in the series is of the form $k_j t^1$, we collect all the coefficients of t^1 and solve the linear system of equations. The second term in the Puiseux series expansion for the cyclic 8-root system, has the form as at the right of (28).

$$\begin{cases} z_0 = t \\ z_1 = -I + c_1 t \\ z_2 = \frac{-1}{2} - \frac{I}{2} + c_2 t \\ z_3 = -1 + c_3 t \\ z_4 = 1 + I + c_4 t \\ z_5 = \frac{1}{2} + \frac{I}{2} + c_5 t \\ z_6 = I + c_6 t \\ z_7 = (-1 - I) + c_7 t \end{cases} \qquad \begin{cases} z_0 = t \\ z_1 = -I + (-1 - I)t \\ z_2 = \frac{-1}{2} - \frac{I}{2} + \frac{1}{2}t \\ z_3 = -1 \\ z_4 = 1 + I - t \\ z_5 = \frac{1}{2} + \frac{I}{2} - \frac{1}{2}t \\ z_6 = I + (1 + I)t \\ z_7 = (-1 - I) + t \end{cases} \tag{28}$$

Because of the regularity of the solution of the initial form system and the second term of the Puiseux series, we have a symbolic-numeric representation of a quadratic solution curve.

If we place the same pretropism in another row in the unimodular matrix, then we can develop the same curve starting at a different coordinate plane. This move is useful if the solution curve would not be in general position with respect to the first coordinate plane. For symmetric polynomial systems, we apply the permutations to the pretropism, the initial form systems, and its solutions to find Puiseux series for different solution curves, related to the generating pretropism by symmetry.

Also for the pretropism $\mathbf{v} = (1, -1, 1, -1, 1, -1, 1, -1)$, the coordinate transformation is given by the unimodular matrix M equal to the identity matrix, except for its first row $\mathbf{v}$. The coordinate transformation $\mathbf{x} = \mathbf{z}^M$ yields $x_0 = z_0$, $x_1 = z_1/z_0$, $x_2 = z_0 z_2$, $x_3 = z_3/z_0$, $x_4 = z_0 z_4$, $x_5 = z_5/z_0$, $x_6 = z_0 z_6$, $x_7 = z_7/z_0$. Applying the coordinate transformation to $\text{in}_{\mathbf{v}}(C_8)(\mathbf{x})$ gives

$$\text{in}_{\mathbf{v}}(C_8)(\mathbf{x} = \mathbf{z}^M) = \begin{cases} z_1 + z_3 + z_5 + z_7 = 0 \\ z_1 z_2 + z_2 z_3 + z_3 z_4 + z_4 z_5 + z_5 z_6 + z_6 z_7 + z_1 + z_7 = 0 \\ z_1 z_2 z_3 + z_3 z_4 z_5 + z_5 z_6 z_7 + z_1 z_7 = 0 \\ z_1 z_2 z_3 z_4 + z_2 z_3 z_4 z_5 + z_3 z_4 z_5 z_6 + z_4 z_5 z_6 z_7 + z_1 z_2 z_3 \\ + z_1 z_2 z_7 + z_1 z_6 z_7 + z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 + z_3 z_4 z_5 z_6 z_7 + z_1 z_2 z_3 z_7 + z_1 z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 z_6 + z_2 z_3 z_4 z_5 z_6 z_7 + z_1 z_2 z_3 z_4 z_5 + z_1 z_2 z_3 z_4 z_7 \\ + z_1 z_2 z_3 z_6 z_7 + z_1 z_2 z_5 z_6 z_7 + z_1 z_4 z_5 z_6 z_7 + z_3 z_4 z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 z_6 z_7 + z_1 z_2 z_3 z_4 z_5 z_7 + z_1 z_2 z_3 z_5 z_6 z_7 + z_1 z_3 z_4 z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 z_6 z_7 - 1 = 0 \end{cases} \tag{29}$$

The initial form system (29) has 72 solutions. Among the 72 solutions, a solution of the form

$$z_0 = t, \; z_1 = -1, \; z_2 = I, \; z_3 = -I, \; z_4 = -1, \; z_5 = 1, \; z_6 = -I, \; z_7 = I, \tag{30}$$

here expressed in the original coordinates,

$$x_0 = t,\ x_1 = -1/t,\ x_2 = It,\ x_3 = -I/t,\ x_4 = -t,\ x_5 = 1/t,\ x_6 = -It,\ x_7 = I/t \tag{31}$$

satisfies the cyclic 8-roots entirely. Applying the cyclic permutation of this solution set we can obtain the remaining 7 solution sets, which also satisfy the cyclic 8-roots system.

In [56], a formula for the degree of the curve was derived, based on the coordinates of the tropism and the number of initial roots for the same tropism. We apply this formula and obtain 144 as the known degree of the space curve of the one dimensional solution set, see Table 3.

$(1, -1, 1, -1, 1, -1, 1, -1)$	$8 \times 2 = 16$
$(1, -1, 0, 1, 0, 0, -1, 0) \rightarrow (1, 0, 0, -1, 0, 1, -1, 0)$	$8 \times 2 + 8 \times 2 = 32$
$(1, 0, -1, 0, 0, 1, 0, -1) \rightarrow (1, 0, -1, 1, 0, -1, 0, 0)$	$8 \times 2 + 8 \times 2 = 32$
$(1, 0, -1, 1, 0, -1, 0, 0) \rightarrow (1, 0, -1, 0, 0, 1, 0, -1)$	$8 \times 2 + 8 \times 2 = 32$
$(1, 0, 0, -1, 0, 1, -1, 0) \rightarrow (1, -1, 0, 1, 0, 0, -1, 0)$	$8 \times 2 + 8 \times 2 = 32$
	TOTAL = 144

Table 3: Tropisms, cyclic permutations, and degrees for the cyclic 8 solution curve.

Using the same polyhedral method we can find all the isolated solutions of the cyclic 8-roots system. We conclude this subsection with some empirical observations on the time complexity. In the direction $(1, -1, 0, 1, 0, 0, -1, 0)$, there is a second term in the Puiseux series as for the 40 solutions of the initial form system, there is no first term that satisfies the entire cyclic 8-roots system. Continuing to construct the second term, the total time required is 35.5 seconds, which includes 28 milliseconds that PHCpack needed to solve the initial form system. For $(1, -1, 1, -1, 1, -1, 1, -1)$ there is no second term in the Puiseux series as the first term satisfies the entire system. Hence, the procedure for construction and computation of the second term does not run. It takes PHCpack 12 seconds to solve the initial form system, whose solution set consists of 509 solutions. Determining that there is no second term for the 509 solutions, takes 199 seconds. Given their numbers of solutions, the ratio for time comparison is given by $\frac{509}{40} = 12.725$. However, given that for tropisms $(1, -1, 0, 1, 0, 0, -1, 0)$ the procedure for construction and computation of the second term does run, unlike for tropism $(1, -1, 1, -1, 1, -1, 1, -1)$, the ratio for time comparison is not precise enough. A more accurate ratio for comparison is $\frac{199}{35} \approx 5.686$.

5.3 Cyclic 12-roots

The generating solutions to the quadratic space curve solutions of the cyclic 12-roots problem are in Table 4. As the result in the Table 4 is given in the transformed coordinates, we return the solutions to the original coordinates. For any solution generator $(r_1, r_2, \ldots, r_{11})$ in Table 4:

$$\begin{aligned} &z_0 = t,\ z_1 = r_1,\ z_2 = r_2,\ z_3 = r_3,\ z_4 = r_4,\ z_5 = r_5, \\ &z_6 = r_6,\ z_7 = r_7,\ z_8 = r_8,\ z_9 = r_9,\ z_{10} = r_{10},\ z_{11} = r_{11} \end{aligned} \tag{32}$$

and turning to the original coordinates we obtain

$$\begin{aligned} &x_0 = t,\ x_1 = r_1/t,\ x_2 = r_2 t,\ x_3 = r_3/t,\ x_4 = r_4 t,\ x_5 = r_5/t \\ &x_6 = r_6 t,\ x_7 = r_7/t,\ x_8 = r_8 t,\ x_9 = r_9/t,\ x_{10} = r_{10} t,\ x_{11} = r_{11}/t \end{aligned} \tag{33}$$

Application of the degree formula of [56] shows that all space curves are quadrics. Compared to [46], we arrive at this result without the application of any factorization methods.

6 Concluding Remarks

Inspired by an effective proof of the fundamental theorem of tropical algebraic geometry, we outlined in this paper a polyhedral method to compute Puiseux series expansions for solution curves of polynomial systems. The main advantage of the new approach is the capability to exploit permutation symmetry. For our experiments, we relied on cddlib and Gfan for the pretropisms, the blackbox solver of PHCpack for solving the initial form systems, and Sage for the manipulations of the Puiseux series.

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z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}
$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$
$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	1	1	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	-1	-1	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$
$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}$
$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	-1	-1	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	1	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	1
$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2}$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	1	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$
1	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	-1	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	1	-1	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$
$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$
$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	1	-1	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	1	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	-1
1	-1	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	-1	1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$
$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2}$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$
$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	-1	$\frac{1}{2}$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1
$-\frac{1}{2}$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	1	-1	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	-1
$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$
$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$
$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	-1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	-1	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	1	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1
$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$
1	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	-1	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$
-1	-1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$
-1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	-1	1	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	-1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$
$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	$-\frac{1}{2}$	-1	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$

Table 4: Generators of the roots of the initial form system $\text{in}_{\mathbf{v}}(C_{12})(\mathbf{x}) = \mathbf{0}$ with the tropism $\mathbf{v} = (+1, -1, +1, -1, +1, -1, +1, -1, +1, -1, +1, -1)$ in the transformed $\mathbf{z}$ coordinates. Every solution defines a solution curve of the cyclic 12-roots system.