Homework 15 solutions, Math 446, professor Agol, winter 2002

4.1.6.1 If $\mathcal{M}_1$ and $\mathcal{M}_2$ are n-dimensional manifolds, the *connected sum* $\mathcal{M}_1 \# \mathcal{M}_2$ is constructed by removing a small open n-ball $\mathcal{E}_i$ from each $\mathcal{M}_i$ and then identifying $(\mathcal{M}_1 - \mathcal{E}_1)$ and $(\mathcal{M}_2 - \mathcal{E}_2)$ along their boundary $(n-1)$-spheres. When $n \geq 3$ show that $\pi_1(\mathcal{M}_1 \# \mathcal{M}_2) = \pi_1(\mathcal{M}_1) * \pi_1(\mathcal{M}_2)$.

Actually, the phrasing of the connect sum should be more careful. One should take a closed embedded 3-ball $B_i \subset \mathcal{M}_i$, which we may identify homeomorphically with $B_1(0) = \{x \in \mathbb{R}^n, |x| \leq 1\} \subset \mathbb{R}^n$, and then remove the subset corresponding to $\mathcal{E}_i \cong int(B_{\frac{1}{2}}(0)) = \{x \in \mathbb{R}^n, |x| < \frac{1}{2}\} \subset B_i$. Then we glue the boundaries $\partial \mathcal{E}_i \cong \{x \in B_i, |x| = \frac{1}{2}\}$ together by a homeomorphism (there are actually two ways to do this, orientation preserving or reversing, but it won't affect our computation) to get the manifold $\mathcal{M}_1 \# \mathcal{M}_2$. Let $A_i = int(\mathcal{M}_i - \mathcal{E}_i)$, then $int(B_i - \mathcal{E}_i) = int(B_i) \cap A_i$. We have $int(B_i - \mathcal{E}_i) \cong \{x \in \mathbb{R}^n | \frac{1}{2} < |x| < 1\} \simeq S^{n-1}$, so $\pi_1(int(B_i - \mathcal{E}_i)) = 1$, and $\pi_1(int B_i) = 1$. So we have the pushout diagram:

$$\begin{array}{ccc} 1 = \pi_1(A_i \cap int(B_i)) & \longrightarrow & \pi_1(A_i) \\ \downarrow & & \downarrow \\ 1 = \pi_1(int(B_i)) & \longrightarrow & \pi_1(\mathcal{M}_i), \end{array}$$

so we conclude that $\pi_1(A_i) = \pi_1(\mathcal{M}_i)$. Now, we may take $\mathcal{A} = int(B_1) - \mathcal{E}_1 \cup A_2$, and $\mathcal{B} = int(B_2) - \mathcal{E}_2 \cup A_1$. Then $\mathcal{M}_1 \# \mathcal{M}_2 = \mathcal{A} \cup \mathcal{B}$, and $\mathcal{A} \cap \mathcal{B}$ deformation retracts to S^{n-1}. Also, $\mathcal{A} \simeq A_2$, $\mathcal{B} \simeq A_1$. We have the pushout diagram:

$$\begin{array}{ccc} 1 = \pi_1(\mathcal{A} \cap \mathcal{B}) & \longrightarrow & \pi_1(\mathcal{A}) = \pi_1(\mathcal{M}_2) \\ \downarrow & & \downarrow \\ \pi_1(\mathcal{B}) = \pi_1(\mathcal{M}_1) & \longrightarrow & \pi_1(\mathcal{M}_1 \# \mathcal{M}_2) \end{array}$$

Thus, $\pi_1(\mathcal{M}_1 \# \mathcal{M}_2) \cong \pi_1(\mathcal{M}_1) * \pi_1(\mathcal{M}_2)$.

4.2.1.1 Let $\mathcal{C}$ be a connected subcomplex of a triangulated finite connected surface $\mathcal{F}$. If $\mathcal{C} \neq \mathcal{F}$, show that $\mathcal{C}$ collapses onto a graph, so that $\pi_1(\mathcal{C})$ is free.

Induct on the number of triangles of $\mathcal{C}$. Since $\mathcal{C} \neq \mathcal{F}$, there must be some edge and triangle $e \subset T \subset \mathcal{C}$, such that e is contained in no other triangle of $\mathcal{C}$, otherwise $\mathcal{C}$ would be a closed connected surface, contradicting $\mathcal{C} \neq \mathcal{F}$. Thus, we may collapse $\mathcal{C}$ along T, to get a subcomplex $\mathcal{C}' = \mathcal{C} - int(T) - int(e) \simeq \mathcal{C}$, with one fewer triangle. By induction, we eventually find a subgraph $\mathcal{G} \simeq \mathcal{C}$, so $\pi_1(\mathcal{C}) \cong \pi_1(\mathcal{G})$ is free. This argument shows that if $\mathcal{F}$ is a compact connected surface with boundary, then $\pi_1(\mathcal{F})$ is free.

A. The unlink is a disjoint union of two round circles, *e.g.* $L = L_1 \cup L_2 = \{(x,y,z) \in \mathbb{R}^3 \mid (x-2)^2 + y^2 = 1, z = 0\} \cup \{(x,y,z) \in \mathbb{R}^3 \mid (x+2)^2 + y^2 = 1, z = 0\}$. Then show that $\pi_1(\mathbb{R}^3 - L) \cong \mathbb{Z} * \mathbb{Z}$.

We may assume that $\mathbb{R}^3 \subset S^3$, by taking the 1-point Alexandroff compactification. Then $\pi_1(\mathbb{R}^3 - L) \cong \pi_1(S^3 - L)$, since $\mathbb{R}^3 - L = S^3 - L \# \mathbb{R}^3$, and use 4.1.6.1. Then $S^3 - L = (\mathbb{R}^2 \times S^1) \# (\mathbb{R}^2 \times S^1)$. This follows since $S^3 - L_i \cong \mathbb{R}^3 - \{(x,y,z) \in \mathbb{R}^3 | x = y = 0\} \cong \mathbb{R}^2 \times S^1$, as described in class. Then the sphere $\{(x,y,z) \in S^3 \mid (x-2)^2 + y^2 + z^2 = 4\}$ separates L_1 and L_2. So $S^3 - L \cong (S^3 - L_1) \# (S^3 - L_2)$, so $\pi_1(\mathbb{R}^3 - L) \cong \pi_1(S^3 - L) \cong \pi_1((S^3 - L_1) \# (S^3 - L_2)) \cong \mathbb{Z} * \mathbb{Z}$.

B. Prove that the abelianization of the fundamental group of a knot complement is always $\mathbb{Z}$.

Let K be a knot, with generic projection with n crossings, divided up into n bridges $\alpha_1, ..., \alpha_n$, labelled in cyclic order as one traverses K. At the crossing going from strand α_i to α_{i+1} (as in Figure 157), denote the overstrand at this crossing by α_{j_i}. Then the Wirtinger presentation $\pi_1(K) \cong \langle a_1, ..., a_n \mid a_2 = a_{j_1}^{-1} a_1 a_{j_1}, a_3 = a_{j_2}^{-1} a_2 a_{j_2}, ..., a_1 = a_{j_n}^{-1} a_n a_{j_n} \rangle$. In

$$\pi_1(K)^{ab} = \langle a_1, ..., a_n \mid a_{i+1} = a_{j_i}^{-1} a_i a_{j_i}, a_i a_j = a_j a_i, 1 \le i, j \le n \rangle,$$

we then have $a_{i+1} = a_{j_i}^{-1} a_i a_{j_i} = a_i a_{j_i}^{-1} a_{j_i} = a_i$, so $a_1 = a_2 = \cdots = a_n$, so

$$\pi_1(K)^{ab} = \langle a_1, ..., a_n \mid a_{i+1} = a_{j_i}^{-1} a_1 a_{j_i}, a_i a_j = a_j a_i, a_i = a_{i+1}, 1 \le i, j \le n \rangle.$$

But the relators $a_{i+1} = a_{j_i}^{-1} a_i a_{j_i}$ and $a_i a_j = a_j a_i$ follow from the relators $a_i = a_{i+1}$, so we see that $\pi_1(K)^{ab} \cong \langle a_1, ..., a_n | a_{i+1} = a_i, i = 1, ..., n \rangle \cong \mathbb{Z}$ (we may get rid of all but one generator by Tietze moves).