Final Solutions, Math 535, professor Agol, Spring 2007

p. 184, #5.

Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$. Then we have

$$zf(z) + z^2 f(z) = \sum_{k=0}^{\infty} c_k z^{k+1} + \sum_{k=0}^{\infty} c_k z^{k+2} = \sum_{j=1}^{\infty} c_{j-1} z^j + \sum_{n=2}^{\infty} c_{n-2} z^n$$

$$= \sum_{k=2}^{\infty} (c_{k-1} + c_{k-2}) z^k = \sum_{k=2}^{\infty} c_k z^k = f(z) - z.$$

Solving for $f(z)$, we have

$$f(z) = \frac{z}{1 - z - z^2}.$$

Now, the polynomial $z^2 + z - 1 = (z - (-1 + \sqrt{5})/2)(z - (-1 - \sqrt{5})/2) = (z - 1/\varphi)(z + \varphi)$, where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio. We perform the partial fraction decomposition using the method of residues to obtain

$$f(z) = \frac{Res_{z=-\phi} f(z)}{z + \phi} + \frac{Res_{z=1/\phi} f(z)}{z - 1/\phi} = -\frac{-\varphi}{(-\phi - 1/\phi)(z + \varphi)} - \frac{1/\varphi}{(1/\phi + \phi)(z - 1/\varphi)}$$

$$= \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \varphi z} - \frac{1}{1 + z/\varphi} \right) = \frac{1}{\sqrt{5}} \sum_{k=0}^{\infty} (\varphi^k - (-1/\varphi)^k) z^k.$$

Thus, we obtain the identity

$$c_k = \frac{1}{\sqrt{5}} (\varphi^k - (-1/\varphi)^k).$$

p. 186, #3 Suppose that

$$f(z) = a(z - z_0)^m + \cdots ,$$

where $a \neq 0$. Then we have

$$f'(z) = am(z - z_0)^{m-1} + \cdots ,$$

$$f''(z) = am(m-1)(z - z_0)^{m-2} + \cdots ,$$

$$f'''(z) = am(m-1)(m-2)(z - z_0)^{m-3} | \cdots .$$

Thus,

$$f'''(z)/f'(z) = (m-1)(m-2)(z - z_0)^{-2} + \cdots$$

and

$$f''(z)/f'(z) = (m-1)(z - z_0)^{-1} + \cdots .$$

So we have

$$\{f, z\} = (m-1)(m-2)(z - z_0)^{-2} - \frac{3}{2}(m-1)^2 (z - z_0)^{-2} + \cdots = \frac{1}{2}(1 - m^2)(z - z_0)^{-2} + \cdots .$$

p. 227, # 1

Let $\Omega \subset \mathbb{C}$ be a region, and let $\mathcal{F}$ be the family of functions defined in Ω with positive real part. Consider the family of functions $\mathcal{G} = \{1/(1 + f) | f \in \mathcal{F}\}$. Then for $g \in \mathcal{G}$, $|1 + f| \geq |1 + Re(f)| \geq 1$, and therefore $0 < |g(z)| \leq 1$. By theorem 15, $\mathcal{G}$ is a normal family with respect to $\mathbb{C}$. Thus, given any sequence $\{f_n\} \subset \mathcal{F}$, we have a subsequence $\{n_k\}$ such that $\{g_{n_k}\} = \{1/(1+f_{n_k})\}$ converges to a function g uniformly on compact subsets of Ω. By Hurwitz's theorem (Thm. 2, Ch. 5), either the limit $g \equiv 0$ identically on Ω, or g is nowhere zero, since $g_{n_k} \neq 0$. If $g \equiv 0$, then it must be the case that $f_{n_k} \to \infty$ uniformly on compact subsets.

Assume now that $g(z) \neq 0$, $z \in \Omega$. Then $f_{n_k} \to f(z) = 1/g(z) - 1$, and therefore $\mathcal{F}$ is normal.

Let $E_k \subset \Omega$ be the compact subsets constructed on p. 220. Consider $\mathcal{F}' = \{f \in \mathcal{F} | |f(z)| < k, z \in E_k\}$. Then $\mathcal{F}'$ is a locally bounded subset of $\mathcal{F}$.

1. Classify twice punctured simply-connected regions in $\mathbb{C}$. That is, suppose $\Omega_i = \Omega_i' - \{w_i, z_i\}$, $i = 1, 2$, $w_i, z_i \in \Omega_i'$, $w_i \neq z_i$, where Ω_i' is simply-connected. Then Ω_1 and Ω_2 are equivalent if there exists a bijective analytic map $f : \Omega_1' \to \Omega_2'$, such that $f(w_1) = w_2, f(z_1) = z_2$. Find a canonical representative for each equivalence class, and show that the equivalence classes are distinct.

   If $\Omega_1 = \mathbb{C}$, then we may take a linear transformation sending $\{w_1, z_1\}$ to $\{0, 1\}$, for example $(z - w_1)/(z_1 - w_1)$.

   If $\Omega_1 \subset \mathbb{C}$, $\Omega_1 \neq \mathbb{C}$, then by the Riemann mapping theorem, there exists a map $f : \Omega_1 \to B_1(0)$, such that $f(w_1) = 0$. Composing with a rotation of the form $e^{i\theta}$, we may assume that $f(z_1) = t$, where $0 < t < 1$. We claim that $\{B_1(0), 0, t\}, t \in (0, 1)$ and $\{\mathbb{C}, 0, 1\}$ form a complete set of distinct representatives of the equivalence classes of twice-punctured simply-connected domains in $\mathbb{C}$.

   First, notice that $\mathbb{C}$ is not holomorphically equivalent to $B_1(0)$, so these form distinct classes. Now, suppose that there is a bijective analytic function $f : B_1(0) \to B_1(0)$, $f(0) = 0$, $f(t_1) = t_2$, $0 < t_1 < t_2$. By homework problem 5, p. 136, we must have $f = az$, where $|a| = 1$. But this is impossible, since otherwise $|f(t_1)| = |at_1| = t_1 < t_2 = f(t_1)$. So these twice punctured regions must be pairwise distinct.

2. Compute

   $$I_t = \int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 e^{itx} dx$$

   for real t.

   First, note that $\sin^2 z e^{itz}/z^2$ is an entire analytic function. Thus, the integral over $[-A, A]$ equals that over the path γ_A obtained by going from $-A$ to -1 along the real axis, from -1 to 1 along the lower unit circle, and along 1 to A along the real axis. Since γ_A avoids $z = 0$, we may use the identity $\sin z = (e^{iz} - e^{-iz})/2i$ to see that the integral may be given by

   $$I_t = -\frac{1}{4} \int_{\gamma_A} (e^{2iz} - 2 + e^{-2iz}) e^{itz}/z^2 dz,$$

   since all of these integrals are absolutely convergent.

   Write

   $$\phi_A(s) = \int_{\gamma_A} \frac{e^{isz}}{z^2} dz.$$

   The we may express the integral as

   $$\lim_{A\to\infty} -\frac{1}{4}(\phi_A(t+2) - 2\phi_A(t) + \phi_A(t-2)).$$

   We will compute $\lim\limits_{A\to\infty} \phi_A(s)$ differently depending on the sign of s.

Let μ_A be the semicircle connecting $-A$ to A in the upper half plane. For $s > 0$, we have $|e^{is(x+iy)}| = e^{-sy} \leq 1$ for $y \geq 0$, so we have the limit of the following integrals given by

$$\lim_{A\to\infty} |\int_{\mu_A} e^{isz}/z^2 dz| \leq \lim_{A\to\infty} \int_{\mu_A} |\frac{e^{isz}}{z^2}||dz|$$

$$\leq \lim_{A\to\infty} \int_0^\pi \frac{e^{-sy}}{A^2} A d\theta \leq \lim_{A\to\infty} \pi/A = 0.$$

But we have $\int_{\gamma_A - \eta_A} e^{isz}/z^2 dz = 2\pi i Res_{z=0} e^{isz}/z^2 = 2\pi i(is) = -2\pi s$. So we have

$$\lim_{A\to\infty} \phi_A(s) = -2\pi s.$$

If $s = 0$, then integrating we have

$$\lim_{A\to\infty} \phi_A(s) = \lim_{A\to\infty} \int_{\gamma_A} \frac{dz}{z^2} = \lim_{A\to\infty} \left[\frac{-1}{z}\right]_{-A}^{A} = 0.$$

If $s < 0$, then we use the lower semicircle η_A connecting $-A$ to A. Similar reasoning to the above shows that

$$\lim_{A\to\infty} \int_{\eta_A} e^{isz}/z^2 dz = 0.$$

Thus, $\lim\limits_{A\to\infty} \phi_A(s) = 0$ for $s \leq 0$.

Putting this together, we have

for $|t| > 2$, $I_t = 0$; for $|t| \leq 2$, we have $I_t = \pi/2(2 - |t|)$.

3. (a) Show that

$$f(z) = \frac{1}{e^z - 1} = \sum_{k=-1}^{\infty} B_{k+1} z^k/(k+1)!,$$

where $B_0 = 1$, $B_1 = -\frac{1}{2}$, and $B_{2k+1} = 0$ for $k \in \mathbb{N}$. Prove that $\overline{\lim\limits_{k\to\infty}} |B_{k+1}/(k+1)!|^{1/k} = 1/2\pi$, and compute B_2, B_4, B_6.

We see that

$$\frac{1}{e^z - 1} + \frac{1}{2} = \frac{e^z + 1}{2(e^z - 1)}.$$

Also, we have

$$\frac{1}{e^{-z} - 1} + \frac{1}{2} = \frac{e^{-z} + 1}{2(e^{-z} - 1)} = \frac{1 + e^z}{2(1 - e^z)} = -\frac{1}{e^z - 1} - \frac{1}{2}.$$

Thus, $1/(e^z - 1) + \frac{1}{2}$ is an odd function of z, and therefore the even coefficients of the Laurent series about $z = 0$ vanish, that is $B_{2k+1} = 0$ for $k > 0$.

We also see from $e^z - 1 = z + o(z)$ that $Res_{z=0} 1/(e^z - 1) = 1$. Now, the function $\frac{1}{e^z - 1}$ is an entire meromorphic function, and has poles at $z = 2\pi i n$, $n \in \mathbb{Z}$. Thus, the radius of convergence of the Laurent series about $z = 0$ is 2π, since it is defined on the annulus $0 < |z| < 2\pi$ by Ch. 5, section 1.3. Then apply Hadamard's theorem to conclude that $\overline{\lim\limits_{k\to\infty}} |B_{k+1}/(k+1)!|^{1/k} = 1/2\pi$.

We may compute that $B_2 = 1/6, B_4 = -1/30$, and $B_6 = 1/42$ by long division of $z + z^2/2 + z^3/6 + z^4/24 + z^5/120 + z^6/720$ into 1. A slightly simpler method is to compute the coefficients inductively using:

$$1 = \left(\sum_{j=1}^{\infty} \frac{z^j}{j!}\right)\left(\sum_{k=-1}^{\infty} \frac{B_{k+1}}{(k+1)!} z^k\right) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} \sum_{j=1}^{n+1} B_{n-j+1} \binom{n+1}{j},$$

using the change of indices $n = k + j$. Then comparing coefficients, we see that $B_0 = 1$, and $0 = \sum_{j=1}^{n+1} B_{n-j+1}\binom{n+1}{j}$, $n \geq 1$. One may then compute the coefficients inductively, simplifying the computation using the fact that $B_{2k+1} = 0$.

(b) Compute a Taylor series about $z = 0$ for

$$\frac{e^{mz} - 1}{e^z - 1}$$

two different ways, where $m \in \mathbb{N}$. Compare coefficients, to show that

$$\sum_{k=0}^{m-1} k^n = \frac{1}{n+1} \sum_{j=1}^{n+1} \binom{n+1}{j} m^j B_{n+1-j}.$$

Find the polynomial expression for $\sum_{k=0}^{m-1} k^6$.

Using the sum of a geometric series, we see that

$$\frac{e^{mz} - 1}{e^z - 1} = \sum_{k=0}^{m-1} e^{kz} = \sum_{k=0}^{m-1} \sum_{n=0}^{\infty} (kz)^n/n! = \sum_{n=0}^{\infty} z^n/n! \left(\sum_{k=0}^{m-1} k^n\right).$$

On the other hand, we have

$$\frac{e^{mz} - 1}{e^z - 1} = \left(\sum_{j=1}^{\infty} (mz)^j/j!\right)\left(\sum_{k=-1}^{\infty} B_{k+1} z^k/(k+1)!\right)$$

$$= \sum_{n=0}^{\infty} \sum_{k+j=n} (mz)^j/j! B_{k+1} z^k/(k+1)! = \sum_{n=0}^{\infty} z^n/n! \frac{1}{n+1} \sum_{j=1}^{n+1} B_{n-j+1} m^j \binom{n+1}{j}.$$

Comparing coefficients of $z^n/n!$ gives the desired formula.

We have

$$\sum_{k=0}^{m-1} k^6 = \frac{1}{7} \sum_{j=1}^{7} B_{7-j} m^j \binom{7}{j}$$

$$= \frac{1}{7}\left(B_6 \binom{7}{1} m + B_4 \binom{7}{3} m^3 + B_2 \binom{7}{5} m^5 + B_1 \binom{7}{6} m^6 + B_0 \binom{7}{7} m^7\right)$$

$$= \frac{1}{7} m^7 - 1/2m^6 + 1/2m^5 - 1/6m^3 + 1/42m.$$

4. Show that if a, b, c lie on the unit circle, and $a + b + c = 0$, then they form the vertices of an equilateral triangle.

   We have $|a| = |b| = |c| = 1$, so $|a|^2 = |b|^2 = |c|^2 = 1$. Also, $a+b = -c$, so $|a+b|^2 = |-c|^2 = 1$. From equation (8), p. 8, we have

   $$1 + |a - b|^2 = |a + b|^2 + |a - b|^2 = 2(|a|^2 + |b|^2) = 4,$$

   so $|a - b|^2 = 3$, and $|a - b| = \sqrt{3}$. By symmetry of the equation, $|b - c| = |c - a| = \sqrt{3}$, so a, b, c form the vertices of an equilateral triangle.