THURSTON'S CONGRUENCE LINK

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Klein's quartic curve may be described as the Riemann surface obtained by taking the quotient of $\mathbb{H}^2$ by the (principal congruence) subgroup $\Gamma(7) = ker\,\{\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/7\mathbb{Z})\}$, and filling points in the cusps (punctures) to get a closed surface (although the punctured surface is sometimes also referred to as Klein's quartic). It has a cell decomposition by 24 heptagons, centered at each cusp coming from the Epstein-Penner-Ford domain of $\mathbb{H}^2/\Gamma(7)$. Each heptagon is fixed by a rotation of order 7, which also preserves two other heptagons, giving a grouping of the heptagons into 8 classes which are preserved by the symmetries of the surface. Rotating one heptagon 1/7th of a turn corresponds to rotating one other 2/7ths, and the third 4/7ths. During a lecture at MSRI on Klein's quartic commemorating the installation of Helaman Ferguson's sculpture "8-fold way" [2], Thurston noticed that the group of symmetries preserving each class of heptagons is the same as the group of symmetries of the triangulation of the torus whose 1-skeleton is the complete graph on 7 vertices. Thurston wondered if there might be a way of relating these two symmetries, and found a hyperbolic 3-manifold with 8 cusps which gives such a relation.

The cusps of $S = \mathbb{H}^2/\Gamma(7)$ correspond to the orbits of $\Gamma(7)$ acting on $\hat{\mathbb{Q}} = \mathbb{Q} \cup \infty$. These correspond to

$$\{\pm(a,b) \in \mathbb{Z}^2 |\, \gcd(a,b,7) = 1\} (\mathrm{mod} 7).$$

Clearly, there are $(7^2-1)/2 = 24$ such cusps, since they correspond to $\pm(a,b) \in (\mathbb{F}_7^2 - \{(0,0)\})/\{\pm 1\}$, where $\mathbb{F}_7 = \mathbb{Z}/7\mathbb{Z}$. The matrices fixing $\infty = (1,0)$ are the upper triangular matrices in $\mathrm{SL}_2(\mathbb{Z})$ with trace ± 2. These matrices also fix the cusps corresponding to the orbits of $(2,7)$ and $(3,7)$ by $\Gamma(7)$ (remember, these are taken $(\mathrm{mod} 7)$, so these correspond to $(2,0)$ and $(3,0)(\mathrm{mod} 7)$ respectively, which are clearly fixed by upper triangular matrices $(\mathrm{mod} 7)$). The matrices in

$\mathrm{PGL}_2(\mathbb{Z})$ which are upper triangular (mod7) permute the $\Gamma(7)$ orbits of these three cusps. The cusps are divided into classes corresponding to elements of $\mathbb{F}_7\mathbb{P}^1 = (\mathbb{F}_7^2 - \{(0,0)\})/\mathbb{F}_7^\times$, which corresponds to $\mathbb{F}_7 \cup \infty = \{(a,1), a \in \mathbb{F}_7\} \cup \{(1,0)\}$, and therefore has 8 classes.

If we let $\zeta = e^{i\pi/3} = (1+\sqrt{-3})/2$, the ring of integers in the field $\mathbb{Q}(\zeta)$ is $\mathbb{Z}[\zeta]$, and the norm in this field is

$$\mathcal{N}(a+b\zeta) = (a+b\zeta)(a+b\overline{\zeta}) = a^2+ab+b^2 = \#\{\mathbb{Z}[\zeta]/(a+b\zeta)\mathbb{Z}[\zeta]\}.$$

So the element $2+\zeta$ has norm 7 in this field. Let $\mathrm{GL}(2,\mathbb{Z}[\zeta])$ consist of the 2×2 matrices whose determinant is invertible in $\mathbb{Z}[\zeta]$, and $\mathrm{PGL}(2,\mathbb{Z}[\zeta])$ is the quotient by the center. Thus, if we take

$$\Delta(2+\zeta) = \ker[\mathrm{PGL}(2,\mathbb{Z}[\zeta]) \to \mathrm{PGL}(2,\mathbb{Z}[\zeta]/(2+\zeta)\mathbb{Z}[\zeta])],$$

we will have $\Gamma(7) \leq \Delta(2+\zeta)$, since $7\mathbb{Z} = (2+\overline{\zeta})(2+\zeta)\mathbb{Z} \subset (2+\zeta)\mathbb{Z}[\zeta]$. The cusps of $\mathbb{H}^3/\Delta(2+\zeta)$ correspond to the orbits of $\Delta(2+\zeta)$ acting on $\widehat{\mathbb{Q}(\zeta)} = \mathbb{Q}(\zeta)\cup\infty$. These correspond to

$$\{\zeta^k \cdot (a,b) \in \mathbb{Z}[\zeta]^2 \mid \gcd(a,b,2+\zeta)=1\}(\mathrm{mod}(2+\zeta)),$$

since any element of $\mathbb{Q}(\zeta)$ may be expressed as a fraction $\frac{a}{b}$, $a,b \in \mathbb{Z}[\zeta]$, up to multiplication by units ζ^k. But $\Delta(2+\zeta)$ preserves the congruency classes of $(a,b)(\mathrm{mod}(2+\zeta))$. Thus, we see that the number of cusps is $\mathbb{F}_7\mathbb{P}^1$, which has 8 elements, and thus $\mathbb{H}^3/\Delta(2+\zeta)$ has only 8 cusps. This means that under the map $\mathbb{H}^2/\Gamma(7) \to \mathbb{H}^3/\Delta(2+\zeta)$, the 24 cusps of $\mathbb{H}^2/\Gamma(7)$ must be identified to 8 cusps of $\mathbb{H}^3/\Delta(2+\zeta)$. Indeed, the cusps $(1,0),(2,0),(3,0)$ all become identified, since for example $-\zeta\cdot(1,0) \equiv (2,0)(\mathrm{mod}(2+\zeta))$.

Consider the lattice $\mathbb{Z}[\zeta] \subset \mathbb{C}$. The orientation preserving isometries G of $\mathbb{C}$ which preserve this lattice are of the form $z \to uz+v$, where $v \in \mathbb{Z}[\zeta]$, and $u \in \mathbb{Z}[\zeta]^\times$ is a unit. Therefore, $u = \zeta^k$, for some $0 \leq k < 6$. Consider the interval $[0,1] \subset \mathbb{C}$, and its orbit $G([0,1])$ under this group of isometries. This is the triangular grid, decomposing $\mathbb{C}$ into equilateral triangles. If we take the ideal $I = (2+\zeta)\mathbb{Z}[\zeta]$, then $T = \mathbb{C}/I$ will be a torus, and $G([0,1])$ projects to an embedding of the complete graph $K_7 \subset T$. $\mathbb{Z}[\zeta]$ embeds in G as a subgroup of translations, and the group G/I gives the group of orientation preserving symmetries of T. Since $\mathbb{Z}[\zeta]/I \cong \mathbb{Z}/7\mathbb{Z}$, the subgroup of G/I generated by translations is generated by the translation $z \to z+1$. But this corresponds to the translation $z \to z + 2\zeta^2 (\mathrm{mod} I)$, since $1 - 2\zeta^2 = -\zeta^2(\zeta+2) \in I$. Similarly, this corresponds to the translation $z \to z + 4\zeta^4 (\mathrm{mod} I)$, since $2\zeta^2 - 4\zeta^4 \in I$.

$\mathrm{PSL}(2,\mathbb{C})$ acts on $\mathbb{H}^3 = \{z + tj, z \in \mathbb{C}, t > 0\} \subset \mathbb{H}$, where $\mathbb{H}$ denotes Hamilton's quaternions. Given $w \in \mathbb{H}^3$, and a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{C})$, the action of A on $\mathbb{H}^3$ is given by $w \to (cw+d)^{-1}(aw+b)$. One may check that this action is well defined, and gives an action of $\mathrm{PGL}(2,\mathbb{C})$, since multiples of the identity matrix act trivially. In this case, the subgroup $\mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{PSL}(2,\mathbb{C})$ preserves the totally geodesic subspace $\mathbb{H}^2 = \{x + tj, x \in \mathbb{R}, t > 0\} \subset \mathbb{H}^3$ preserving orientation. One may easily check that the group $\mathrm{PSL}(2,\mathbb{R}) \cap \Delta(2+\zeta)$ preserving $\mathbb{H}^2$ is precisely $\Gamma(7)$.

Since $\mathbb{Z}[\zeta]/(2+\zeta)\mathbb{Z}[\zeta] \cong \mathbb{Z}/7\mathbb{Z}$, one can check that

$$\mathrm{PSL}(2, \mathbb{Z}[\zeta]/(2+\zeta)\mathbb{Z}[\zeta]) = \mathrm{PSL}(2, \mathbb{Z}/7\mathbb{Z}).$$

Similarly,

$$\mathrm{PGL}(2, \mathbb{Z}[\zeta]/(2+\zeta)\mathbb{Z}[\zeta]) = \mathrm{PGL}(2, \mathbb{Z}/7\mathbb{Z}).$$

$\mathrm{PSL}(2,\mathbb{Z}/7\mathbb{Z})$ is the unique simple group of order 168, which is the group of orientation preserving isometries of the Klein quartic, since by Hurwitz's theorem, the quotient of $\mathbb{H}^2$ by this symmetry group has the minimal area of any orientable hyperbolic orbifold. $\mathrm{PGL}(2,\mathbb{Z}/7\mathbb{Z})$ has order 336, and consists of the full group of isometries of the Klein quartic, since the element $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ acts as a reflection on $\mathbb{H}^2$.

Thurston found an 8 component link in S^3 whose complement is the manifold $\mathbb{H}^3/\Delta(2+\zeta)$, given in figure 18 of [1] (Fig. 1). Of course, the symmetries of the link complement do not extend to S^3, since $\mathrm{PGL}(2,\mathbb{Z}/7\mathbb{Z})$ does not preserve the meridian slopes of the cusps. So in fact, there are many ways to embed the link complement into S^3. The symmetry group of each cusp is the group of upper triangular matrices in $\mathrm{PGL}(2,\mathbb{Z}/7\mathbb{Z})$, which is isomorphic to G/I. The link complement has 8 cusps, and a canonical triangulation dual to the Ford decomposition. Each pair of cusps is connected by a unique edge of the

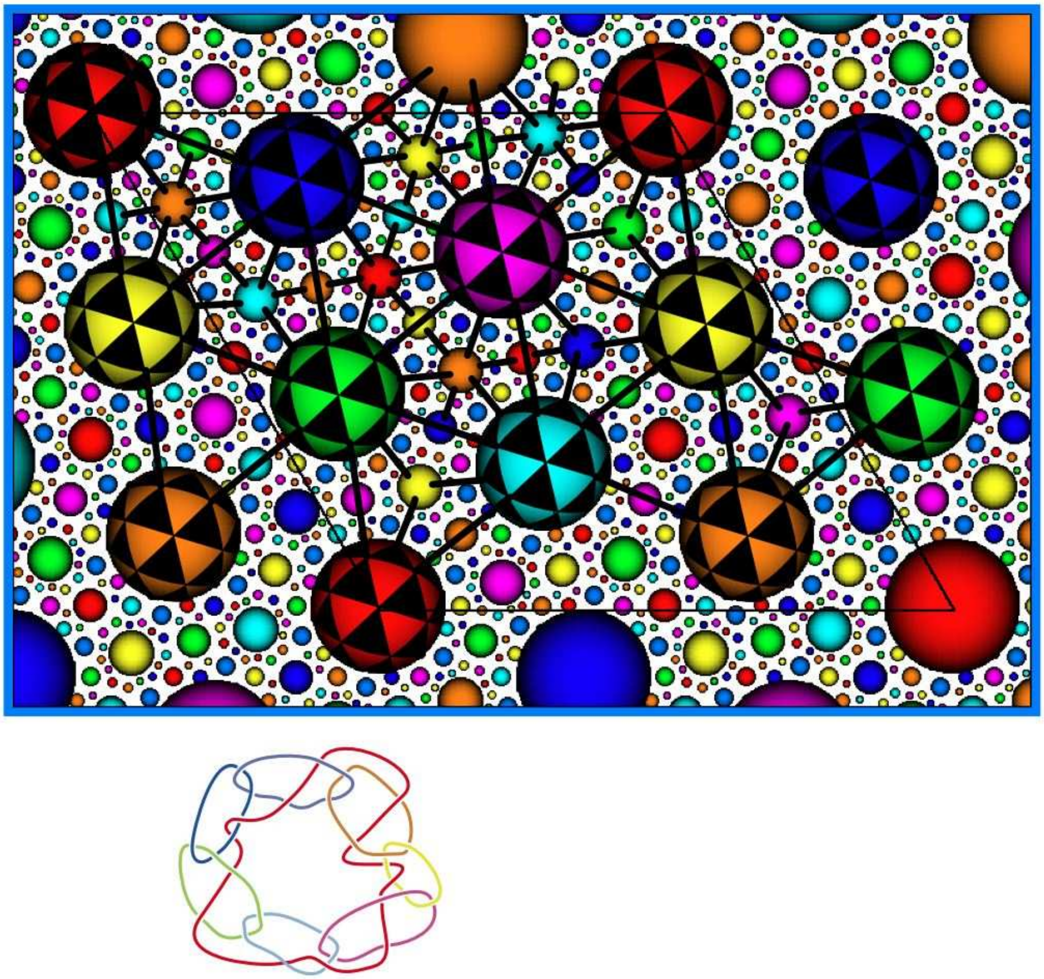

Figure 1. Thurston's congruence link

of the triangulation, giving 28 edges in all. The link of the triangulation of each cusp is a copy of K_7. There are also 28 ideal tetrahedra which meet 6 to an edge (as can be seen from the combinatorics of the links of ideal vertices, see Fig. 1), which have dihedral angles $\pi/3$, and are maximal volume tetrahedra of volume $1.01494... = 3\Lambda(\pi/3)$. The immersed Klein quartic $\mathbb{H}^2/\Gamma(7)$ gives the 2-skeleton of the triangulation. The symmetry given by $z \to z+1$ fixes a cusp of $\mathbb{H}^3/\Delta(2+\zeta)$, and acts as a rotation by $1/7$ on one cusp of the Klein quartic (parallel to $\mathbb{R}$ when lifted to a cusp centered at ∞ in $\mathbb{H}^3$, and by a rotation of $2/7$ on the cusp parallel to $\zeta^2\mathbb{R}$, since it is equivalent to the translation

$z \to 2\zeta^2 \pmod{I}$, and by a rotation of 4/7 on the cusp parallel to $\zeta^4\mathbb{R}$, since it is equivalent to the translation $z \to 4\zeta^4 \pmod{I}$. Interestingly, $\mathbb{H}^3/\Delta(2+\zeta)$ is chiral, even though $\mathbb{H}^3/\mathrm{PGL}(2,\mathbb{Z}[\zeta])$ is amphichiral, and indeed is the double of the minimal volume non-compact hyperbolic Coxeter group. The reflection symmetry does not lift to the congruence cover, since it would have to preserve the immersed Klein quartic surface, which is clearly impossible.

References

[1] William P. Thurston, *How to see 3-manifolds*, Classical Quantum Gravity **15** (1998), no. 9, 2545–2571, Topology of the Universe Conference (Cleveland, OH, 1997).

[2] ———, *The Eightfold Way: a mathematical sculpture by Helaman Ferguson*, The eightfold way, Math. Sci. Res. Inst. Publ., vol. 35, Cambridge Univ. Press, Cambridge, 1999, pp. 1–7.