Eigenvalues and Eigenvectors

A scalar λ is said to be an eigenvalue of a matrix A if

$$\det(A - \lambda I) = 0$$

A vector y is said to be an eigenvector of A, with eigenvalue λ, if

$$Ay = \lambda y$$

Fun Facts

- The trace of A (sum of its diagonal elements) equals the sum of its eigenvalues
- The determinant of A equals the product of its eigenvalues.
- If A is diagonalizable, then the eigenvalues of A form a basis for R^n, so that any vector x can be written as a sum of the eigenvectors $x = \sum_i c_i y_i$, where c_i are coefficients which depend on x.
- $|\lambda| < ||A||_p$ for all p

The key fact related to eigenvalues for this week's material is that an iteration

$$x^{k+1} = Mx^k + f$$

converges when all eigenvalues λ of M, have $|\lambda| < 1$. To see this, decompose x_0 and f into eigenvectors,

$$x_0 = \sum_i c_i y_i \qquad f = \sum_i d_i y_i$$

The trick is that $Mx_0 = M \sum c_i y_i = \sum c_i \lambda_i y_i$. Iterating gives

$$x^k = \sum_i c_i \lambda_i^k y_i + \sum_{j=0}^{k-1} \sum_i d_i \lambda_i^j y_i$$

These sums are finite as $k \to \infty$ if $|\lambda| < 1$.