Final Exam Material

Note: This is not an exhaustive list of problems. Please do not restrict your studying to this review sheet.

1. Consider the parallogram whose vertices are at $(0,0,0), (1,\sqrt{3},0), (1+\sqrt{3},1+\sqrt{3},0)$, and $(\sqrt{3},1,0)$.

   [i] Find all the interior angles in this parallelogram.
   Soln Use $u \cdot v = ||u|| ||v|| \cos(\theta)$, to get $2\sqrt{3} = 4\cos(\theta)$, or $\theta = \pi/6$. This gives the two smaller angles in the parallelogram. The other two angles can be found either by knowing that the angles in a parallelogram add to 2π, or by drawing parallel lines. In either case the larger angles have $\theta = 5\pi/6$.

   [ii] Find the area of this parallelogram, first using vector arithmetic, then using a double integral.
   Soln Area of a parallelogram $= ||u \times v||$. Or we can write three double integrals

   $$\int_0^1 \int_{x/\sqrt{3}}^{\sqrt{3}x} dydx + \int_1^{\sqrt{3}} \int_{x/\sqrt{3}}^{x/\sqrt{3}+\sqrt{3}/2} dydx + \int_{\sqrt{3}}^{1+\sqrt{3}} \int_{\sqrt{3}x-2}^{x/\sqrt{3}+\sqrt{3}/2} dydx$$

   In both cases the area is 2.

2. Find a vector perpendicular to both $u = (1,2,1)$ and $v = (1,4,0)$.
   Soln $n = u \times v = (-4,1,2)$.

3. Find the equation of the plane which contains the the origin and the points $(1,2,1)$ and $(1,4,0)$.
   Soln Recall that any vector in a plane is perpendicular to the normal vector of the plane. The normal vector is $n = (-4,1,2)$. A vector in the plane is $(x-a, y-b, z-c)$ where (a,b,c) is a point in the plane. Choosing $(a,b,c) = (1,2,1)$ the equation for the plane is

   $$(-4,1,2) \cdot (x-1, y-2, z-3) = 0$$

4. Find the equation of the plane $z = x$ in cylindrical, and also, in spherical coordinates.
   Soln Simply plug in the definition $x = r\cos(\theta)$ to get the plane in cylindrical coordinates is $z = r\cos(\theta)$. For spherical, $\rho\cos(\phi) = \rho\cos(\theta)\sin(\phi)$ can be simplified to $1 = \cos(\theta)\tan(\phi)$.

5. Explain why $\mathbf{r}(t) = (t^2+1, t^2, t^4+1)$ is not the same curve as $\mathbf{r}(s) = (s+1, s, s^2+1)$.
Soln Although the substitution $s = t^2$ in the second equation gives the first equation, the range of, for example, x in the first equation is only $x \geq 1$, where in the second x can be all real numbers. Thus the first curve is only a section of the second curve.

6. Show that $\frac{d}{dt}\big(\mathbf{r}(t) \times \mathbf{r}'(t)\big) = \mathbf{r}(t) \times \mathbf{r}''(t)$ for all twice differentiable $\mathbf{r}(t)$.
Soln Apply the product rule for the cross product, $(a \times b)' = (a' \times b + a \times b')$, to get $\frac{d}{dt}\big(\mathbf{r}(t) \times \mathbf{r}'(t)\big) = \big(\mathbf{r}'(t) \times \mathbf{r}'(t) + (\mathbf{r}(t) \times \mathbf{r}''(t)\big)$. Since the cross product of a vector with itself is zero, this gives the desired result.

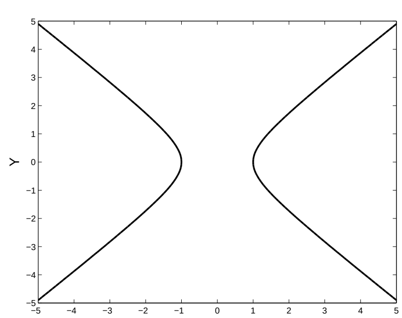
7. Find the tangent vector to $\mathbf{r}(t) = (R\cos(t), R\sin(t), 0)$. Verify that the circumference of a circle is $2\pi R$, by finding the arclength of $\mathbf{r}(t)$ for $0 \leq t \leq 2\pi$.
Soln The tangent vector is $\mathbf{r}'(t) = (-R\sin(t), R\cos(t))$, which has magnitude $||r'(t)|| = R$, so

$$\int_0^{2\pi} ||r'(t)||dt = 2\pi R$$

8. Find the curvature of a circle of radius R, using the above parameterization.
Soln The curvature expressed only in terms of r is

$$\kappa = \frac{||r'(t) \times r''(t)||}{||r'(t)||^3} = \frac{R^2}{R^3} = \frac{1}{R}$$

9. What is the domain of the function $f(x,y) = \ln(x^2 - y^2)$. Find the level curve(s) corresponding to $f(x,y) = 0$.
Soln The domain is the set where the argument of the natural log is positive, or the set of (x,y) with $x^2 > y^2$. The level curve for $f(x,y) = 1$ is $x^2 - y^2 = 1$ is a hyperbola - see below.



10. Is the function $f(x, y) = \arctan(y/x)$ continuous at $(1, 0)$? What about $g(x, y) = \frac{\sqrt{x^2+y^2}}{x+y}$ at $(0, 0)$? And $h(x, y) = \frac{x^2y}{x^2+y^2}$ at $(0, 0)$?
Soln $f(r, \theta) = \theta$ has a jump from 0 to 2π on the positive x-axis, so f is not continuous. For $g(x, y)$, considering lines $y = mx$, we see that the limit must equal $\frac{\sqrt{1+m^2}}{1+m}$. Since the limit depends on direction, it does not exist. For $h(x, y)$ notice that $0 \leq x^2/(x^2 + y^2) \leq 1$, so $0 < h(x, y) < y$. Taking the limit of the inequality, by the squeeze theorem, gives that the $\lim_{(x,y)\to(0,0)} h(x, y) = 0$.

11. Show that $f(x, y) = \sqrt{x^2 + y^2}$ is differentiable near $(1, 0)$ but not near $(0, 0)$.
Soln The gradient is $\nabla f = (\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}})$. For f to be differentiable near a point the gradient must be continuous at that point. To check continuity of a function $g(x, y)$ we require that $g(a, b) = \lim_{(x,y)\to(a,b)} g(x, y)$. Checking lines we can see that neither f_x nor f_y is continuous at $(0, 0)$, so the function is not differentiable. Near $(1, 0)$, f_x and f_y are quotients of non-zero continuous functions, therefore are themselves continuous.

12. Find the tangent plane to $f(x, y) = \exp(-x^2 - y^2)$ at $(0, 0)$.
Soln For $z = f(x, y)$ we can write the tangent plane as $z = \nabla f(a, b) \cdot (x - a, y - b) + f(a, b)$. Here $\nabla f(0, 0) = (0, 0)$ so the tangent plane is $z = 1$.

13. Given a surface $z = f(x, y)$, how is the vector (2 components) ∇f related to the normal vector to the surface (3 components)?
Soln The normal vector to the surface $S(x, y, z) = 0$ is $n = \nabla S = (S_x, S_y, S_z)$. Writing $S = z - f(x, y)$ we get $n = (-f_x, -f_y, 1)$, so the gradient of $f(x, y)$ is proportionate to the first two components of the normal vector to the surface. (Notice we could have written $S = 2z - 2f(x, y) = 0$, so the normal vector is only determined up to a multiplicative constant.

14. In which direction is the function $f(x, y) = \frac{1}{4}x^3 + y^3$ at the point $(x, y) = (1, 2)$ increasing the fastest? decreasing the fastest?
Soln The directional derivative of differentiable function $f(x, y)$ in the direction of a unit vector u is $D_u f = \nabla f \cdot u$. The dot product of two vectors $u \cdot v = ||u||||v|| \cos(\theta)$, where θ is the angle between the vectors. Using this formula with u a unit vector and $v = \nabla f$ we see that $D_u f = ||\nabla f|| \cos(theta)$. In choosing a direction, the only thing we can change is θ so the largest increase occurs when $\theta = 0$, $u = \frac{\nabla f}{||\nabla f||}$, and the largest decrease when $\theta = \pi$, $u = -\frac{\nabla f}{||\nabla f||}$. The directions are $u = \frac{1}{\sqrt{9/16+144}}(3/4, 12)$ and $u = -\frac{1}{\sqrt{9/16+144}}(3/4, 12)$.

15. Consider a coordinate system $u(x, y) = 2x - y$ and $v(x, y) = 5x + 6y$, and the function $f(u, v) = u^2 - v^2$. Find $\frac{\partial f}{\partial x}$ at $x = 1, y = 2$.
Soln Using the chain rule we get $f_x = f_u u_x + f_v v_x$. Here $f_u = 2u$, $f_v = -2v$, $u_x = 2$, $v_x = 5$. Since $u(1, 2) = 0$ and $v(1, 2) = 17$, we get $f_x = (0)(2)+(5)(17) = 85$.

16. Find the global maximum and minimum of $f(x, y) = x^4 - yx + \frac{1}{2}y$ in the domain bounded by $y = x^2, x = 0$, and $y = 1$.
Soln First find interior critical points, $\nabla f = 0$, yielding $(4x^3 - y, -x + 1/2) = 0$, so there is one interior critical point at $(1/2, 1/2)$. Next check the boundaries, on $y = x^2$, $f(x, x^2) = x^4 - x^3 + 1/2x^2$, so $f'(x) = 4x^3 - 3x^2 + x = 0$ gives the critical points in this boundary. This cubic has only one real solution, $x = 0$. Similarly checking the boundary $x = 0$, $f(0, y) = 1/2y$, which has no critical points. Finally, checking $y = 1$, $f(x, 1) = x^4 - x + 1/2$, and here $f'(x) = 4x^3 - 1$, which has a critical point at $x = 1/4^{1/3}$. The points we need to check for the max/min are the intersections of the boundary curves, $(0, 0), (0, 1), (1, 1)$, the interior critical point at $(1/2, 1/2)$ and the boundary critical point at $(1/4^{1/3}, 1)$. The global max is achieved at both $(0, 1)$ and $(1, 1)$, the global min is at $(0, 0)$.

17. Find the closest point to the origin which is on the hyperbola $xy = 2$.
Soln Use Lagrange multipliers, $H(, x, y, \lambda) = x^2 + y^2 + \lambda(xy - 2)$, find where $\nabla H = 0$, or $(2x + \lambda y, 2y + \lambda x, xy - 2) = 0$. Solving for λ in the first equation and substituting into the second we get $y^2 = x^2$ so $y = \pm x$. Substituting this into $xy = 2$, we get $x^2 = 2$ and $-x^2 = 2$. The first gives $x = \pm\sqrt{2}$, the second has no solution, from which we can recover y and get get two points $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$ both of which are a distance 2 from the origin.

18. Calculate $\int\int yx^2\sqrt{1 + x^3}dA$ over the region bounded between $y = x^2 - 1$ and $y = 1 - x^2$.
Soln Observe that this region is even in y, but the integrand is odd in y, so the answer is zero. One may also set it up as $dydx$ and get zero after evaluating the first integral.

19. Calculate $\int_0^1 \int_x^1 e^{-y^2} dydx$.
Soln Change the order of integration, to get $\int_0^1 \int_0^y e^{-y^2} dxdy = \int_0^1 ye^{-y^2} dy = 1/2(1 - e^{-1})$

20. Find the volume of the solid bounded by $z = 0, z = 2x + y + 3, 0 < x < y$, and $0 < y < 1$.
Soln It is natural to write this integral as $dzdxdy$ where it becomes $\int_0^1 \int_0^y \int_0^{2x+y+3} dzdxdy$ or $13/6$.

21. Find the center of mass of the solid in the previous problem.
Soln Use the formula
$$x_c = \int\int\int x\rho(x, y, z)dV / \int\int\int \rho(x, y, z)dV,$$
and similar for y and z, with $\rho = 1$. The denominator is the solution to the previous problem. $x_c = \frac{3/8}{13/6} = \frac{18}{104}$. The process for y_c and z_c is similar.

22. Find the volume of the region inside bounded by $x^2 + y^2 < 1, z < 1 - x^2 - y^2$ and $z > 0$. *Hint, use cylindrical coordinates.*
Soln The volume can be written $\int_0^1 \int_0^{2\pi} \int_0^{1-r^2} rdzd\theta dr = 2\pi \int_0^1 r - r^3 dr = 2\pi(1/2 - 1/4) = \pi/2$

23. Find the volume of the region bounded by $x^2 + y^2 > z^2, x^2 + y^2 + z^2 < 1$, and within the first octant.
Soln Use spherical coordinates to get the integral
$$\int_0^{\pi/2} \int_{\pi/4}^{\pi/2} \int_0^1 \rho^2 \sin(\phi)d\rho d\phi d\theta = \frac{\pi}{6} \int_{\pi/4}^{\pi/2} \sin(\phi)d\phi = \frac{\pi}{6}\big(-\cos(\pi/2)+\cos(\pi/4)\big) = \frac{\sqrt{2}\pi}{12}$$

24. Find the Jacobian of the change of variables $x = u^2 + v^2$, and $y = u^2 - v^2$.
Soln $Jac(\Phi) = x_u y_v - x_v y_u = (2u)(-2v) - (2v)(2u) = -4uv$

25. Evaluate the $\int_C \mathbf{F} \cdot d\mathbf{s}$ for $\mathbf{F} = (x, y^2 x)$ with C the lines which connect $(0, 0)$ to $(1, 0)$ then $(1, 0)$ to $(1, 3)$.
Soln Since F is not conservative we have to do the line integral via parameterizing the curve(s). I use $c_1(t) = (t, 0)$, $c_2(t) = (1, 3t)$, so that the line integral is
$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (t, 0) \cdot (1, 0)dt + \int_0^1 (1, 9t^2) \cdot (0, 3)dt = 1/2 + 9 = \frac{19}{2}$$

26. Show that $\mathbf{F} = (x^3, y^3, z^3)$ is a conservative vector field.
Soln Take the curl of F,
$$\nabla \times F = \begin{pmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^3 & y^3 & z^3 \end{pmatrix} = 0.$$

Since the curl of F is zero everywhere F is conservative. Alternatively one can find a potential, $\phi = \frac{1}{4}(x^4+y^4+z^4)+C$. Since F has a potential F is conservative.

27. Find a potential for $\mathbf{F} = (x^3, y^2, z)$, then evaluate $\int_C \mathbf{F} \cdot d\mathbf{s}$, where C connects $(1, 2, 4)$ to $(3, 1, -1)$.
Soln To find the potential solve the system of equations

$$\phi_x = x^3$$
$$\phi_y = y^2$$
$$\phi_z = z$$

Integrating the first gives $\phi = x^4/4 + f(y, z)$, plugging this into the second gives $f(y, z) = y^3/3 + g(z)$. Plugging this into the third gives $g(z) = z^2/2 + C$, so $\phi = x^4/4 + y^3/3 + z^2/2 + C$.

28. Find the surface area of the cone $z = 1 - \sqrt{x^2 + y^2}$, above $z = 0$.
Soln Since $z = f(x, y)$, we can skip the normal vector stuff, and go straight to $SurfaceArea = \int\int \sqrt{1 + f_x^2 + f_y^2} dA$ The bounds we can recover from observing that the cone is above the z plane only when $x^2 + y^2 < 1$, which inspires us to write this problem in polar coordinates

$$\int\int \sqrt{1 + 4x^2 + 4y^2} dA = \int_0^1 \int_0^{2\pi} r\sqrt{1 + 4r^2} dr d\theta = 2\pi \int_0^1 r\sqrt{1 = 4r^2} dr = \frac{1}{12}(5\sqrt{5} - 1)$$

29. Evaluate the line integral of $\mathbf{F} = (y, -x)$ over the ellipse parameterized $x(t) = 2\cos(t)$ $y(t) = -4\sin(t)$, with $0 \le t \le 2\pi$.
Soln Use Green's Theorem, $\int_C P dx + Q dy = \int\int Q_x - P_y dA$. Here $P = y$ and $Q = -x$, $Q_x - Py = -2$, but our curve is going clockwise, so the answer will be twice the area of the ellipse. Recall ellipses have equation $x^2/a^2 + y^2/b^2 = 1$. Here $a = 2$ and $b = 4$, in which case the area is πab, so the answer is 8π. One could also derive the area of the ellipse with an integral, or just evaluate the line integral without using Green's theorem.