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The generation of capillary-gravity solitary waves by a surface pressure forcing

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Abstract

A weakly nonlinear model is used to study capillary-gravity waves generated by a traveling localized surface pressure distribution. The weakly nonlinear model is a truncation of the potential flow equations in deep water, and includes cubic nonlinear terms. Numerically, solitary waves are shown to be generated by a near-monochromatic, subcritical forcing. The presence of these solitary waves is predicted using a forced nonlinear Schrödinger equation.

AMS Classification: 76B45; 76B25; 76B15

Keywords: Solitary wave; Surface tension; Nonlinear Schrödinger

1. Introduction

In this paper, the generation of capillary-gravity waves by a moving pressure distribution is investigated. We observe that solitary waves can be generated by a subcritical forcing. A family of weakly nonlinear model equations for capillary-gravity water waves is derived from the potential flow equations. Model equations from this family can be characterized by the highest degree in their nonlinear term. We study the forced dynamics of a cubic member of this family.

Understanding forced water waves has been of interest for over a century. Steady flow past an obstacle in the potential flow equations was studied famously by Rayleigh [38] – more recent examples include [33,42]. The dynamic, forced problem has been studied in a number of weakly nonlinear models which include long wave (or shallow water) assumptions, for example the KdV, Benjamin, Benjamin-Ono, and BBM equations [9,6,12,23,24,29,31]. Unforced capillary-gravity wave dynamics have been studied using weakly nonlinear models which do not include a long wave assumption – an overly restrictive assumption for waves at finite wavenumber [2–4,8]. In this paper, the *isotropic* deep water capillary-gravity wave model of [3] is generalized to include cubic terms and a surface pressure distribution. The cubic model Eq. (2.5) is used to study the dynamic problem of flow in the wake of a surface pressure distribution, a simple model for the generation of water waves by wind.

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One feature of capillary-gravity flows which is not present in flows without surface tension is the existence of traveling wavepacket solitary waves. The existence of these waves was first explained by Longuet-Higgins [27]. These waves, and their two-dimensional analogues, have been studied primarily in the context of potential flow, assuming an inviscid fluid, although some recent studies have begun to consider the effect of viscous damping [12,15]. In the inviscid setting, these solitary waves have been predicted with asymptotic arguments [1], rigorously shown to exist in 1D [21] and 2D shallow water [19], and computed numerically [7,10,25,35,36,41]. The dynamics of solitary waves – including stability, collisions, and generation by forcing – have been studied in a number of weakly nonlinear settings. On a two-dimensional fluid, shallow water studies of solitary wave dynamics focus on the KdV equation [5,28,31]. On a three dimensional fluid, shallow water dynamics have been studied with a Benney–Luke system, the KP equations, and the two-dimensional Benjamin equations [8,23,24]. Solitary wave dynamics have also been studied in deep water, both in two and three-dimensions [4,2,3]. The majority of time-dependent simulations have been restricted to *quadratic* weakly-nonlinear models, although some recent progress has been made in direct time-dependent simulations of the potential flow equations [18]. Weakly-nonlinear quadratic models neglect cubic nonlinear terms, which, due to the connection between envelope solitary waves and the cubic nonlinear Schrödinger equation, should play a role in solitary wave dynamics [1,27].

A natural example of surface forcing is that due to wind [11,40], where a number of recent experiments have focused on gravity-capillary wave generation. In 2005 Longuet-Higgins and Zhang experimentally observed transients resembling gravity-capillary solitary waves in the wake of a surface pressure distribution [26]. Similar waves have been observed in experiments in mercury [17] and in a "wind-ruffled" channel [44]. Here we present an inviscid numerical analogue of these experiments, including cubic nonlinear terms, and observe the generation of capillary-gravity solitary waves.

2. Derivation

In this section we derive a weakly nonlinear model for small amplitude water waves which are approximate solutions of the potential flow equation. To begin, recall the potential flow equations for a body of water, of average depth H, displacement η, conservative velocity field $u=\nabla\phi$, which acts under gravity g, atmospheric pressure P, and constant surface tension γ.

$$\Delta_x\phi+\phi_{zz}=0, \qquad -H<z<\eta, \tag{2.1a}$$

$$\phi_z=0, \qquad z=-H, \tag{2.1b}$$

$$\eta_t+\nabla_x\eta\cdot\nabla_x\phi=\phi_z, \qquad z=\eta, \tag{2.1c}$$

$$\phi_t+\frac{1}{2}(\nabla_x\phi)^2+\frac{1}{2}(\phi_z)^2+g\eta+\gamma\nabla\cdot\hat{n}=P, \qquad z=\eta. \tag{2.1d}$$

The operators ∇_x and Δ_x are the gradient and Laplacian in the horizontal spatial coordinates, $\hat{n}$ is the outward unit normal to the free surface. This system has also been reformulated to include the effects of small viscosity and vorticity [15]. We proceed using the approximation of zero viscosity, common for water wave problems. The effect of small viscosity on this problem is a subject of a concurrent investigation. We will henceforth work in dimensionless equations. We choose a characteristic wave height a, a characteristic pressure scale P_0, a lengthscale $\gamma^{1/2}g^{-1/2}$, a timescale $\gamma^{1/4}g^{-3/4}$, and a velocity potential scale $a\gamma^{1/4}g^{1/4}$, natural scales for capillary-gravity waves. The parameter $\epsilon=a/L$ is assumed to be small, the parameter $\delta=P_0/\rho ag$ measures the size of the pressure forcing. In cgs units, $g=981$, cm/s^2 and for an air water interface $\gamma=73.50$, cm^3/s^2 [43]. We are interested in the deep water limit, $H\to\infty$, where the nondimensionalized potential flow equations take the form

$$\Delta_x\phi+\phi_{zz}=0, \qquad -\infty<z<\epsilon\eta, \tag{2.2a}$$

$$\phi_z=0, \qquad z=-\infty, \tag{2.2b}$$

$$\eta_t+\epsilon\nabla_x\eta\cdot\nabla_x\phi=\phi_z, \qquad z=\epsilon\eta, \tag{2.2c}$$

$$\phi_t+\eta+\frac{1}{\epsilon}\nabla\cdot\hat{n}+\epsilon\frac{1}{2}(\nabla_x\phi)^2+\epsilon\frac{1}{2}(\phi_z)^2=\delta P, \qquad z=\epsilon\eta. \tag{2.2d}$$

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Exploiting the smallness of ϵ, we expand the potential and the pressure about the mean depth and write new boundary conditions at $z = 0$.

$$\eta_t - \phi_z + \epsilon\nabla_x \cdot \left(\sum_{n=0}^{\infty}\frac{\epsilon^n}{(n+1)!}\eta^{n+1}\partial_z^n\nabla_x\Phi\right) = 0, \quad \text{at} \quad z = 0, \tag{2.3a}$$

$$\begin{aligned}&\phi_t + \eta - \nabla_x \cdot \left(\frac{\nabla_x\eta}{(1+\epsilon|\nabla_x\eta|^2)^{1/2}}\right) + \sum_{n=1}^{\infty}\frac{\epsilon^n}{n!}\eta^n\partial_z^n\phi_t - \delta\sum_{z=0}^{\infty}\frac{\epsilon^n}{n!}\eta^n\partial_z^n P + \cdots + \frac{\epsilon}{2}\left(\sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}\eta^n\partial_z^n\nabla_x\phi\right)^2\\ &\quad + \frac{\epsilon}{2}\left(\sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}\eta^n\partial_z^n\phi_z\right)^2 = 0, \quad \text{at} \quad z = 0.\end{aligned} \tag{2.3b}$$

Next the z-dependence of the potential can be eliminated by solving Laplace's equation in the lower half space

$$\Phi = \mathcal{F}^{-1}\left\{\mathcal{F}\{\phi(x, y, 0)\}e^{|k|z}\right\}.$$

For numerical simulations, the system must be truncated. At any order of truncation, the system can be written as a pair of first order equations for the potential and the free surface, or a second order equation in terms of one variable only (either the potential or the free surface) by formally eliminating the other [3,32]. Differentiating (2.3a) with respect to time yields

$$\eta_{tt} - \mathcal{L}\Phi_t + \epsilon\nabla_x \cdot \left(\sum_{n=0}^{\infty}\frac{\epsilon^n}{(n+1)!}\eta^{n+1}\mathcal{L}^n\nabla_x\Phi\right)_t = 0, \tag{2.4a}$$

$$\eta_t - \mathcal{L}\Phi + \epsilon\nabla_x \cdot \left(\sum_{n=0}^{\infty}\frac{\epsilon^n}{(n+1)!}\eta^{n+1}\mathcal{L}^n\nabla_x\Phi\right) = 0, \tag{2.4b}$$

$$\begin{aligned}&\Phi_t + \eta - \nabla_x \cdot \left(\frac{\nabla_x\eta}{(1+\epsilon|\nabla_x\eta|^2)^{1/2}}\right) + \sum_{n=1}^{\infty}\frac{\epsilon^n}{n!}\eta^n\mathcal{L}^n\Phi_t - \delta\sum_{z=0}^{\infty}\frac{\epsilon^n}{n!}\eta^n\partial_z^n P + \cdots + \frac{\epsilon}{2}\left(\sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}\eta^n\mathcal{L}^n\nabla_x\Phi\right)^2\\ &\quad + \frac{\epsilon}{2}\left(\sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}\eta^n\mathcal{L}^{n+1}\Phi\right)^2 = 0.\end{aligned} \tag{2.4c}$$

where $\mathcal{L}$ is the operator whose Fourier symbol is $|k|$. For numerical evolution, we consider Eq. (2.4a) as an evolution equation for η with (2.4c) as the recursive definition of Φ_t and (2.4b) as the recursive definition of Φ. Iterating these definitions one can eliminate Φ from Eq. (2.4a). System (2.4) defines approximations of the potential flow equations of arbitrary order; here we consider only the cubic truncation, keeping $O(\epsilon^2, \delta)$ terms. The cubic truncation written as an evolution equation in terms of η on a two dimensional fluid is

$$\eta_{tt} + \Omega^2\eta + \epsilon\mathcal{N}[\eta] + \epsilon^2\mathcal{C}[\eta] = \delta\mathcal{L}P, \tag{2.5}$$

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with

$$\begin{aligned}
\mathcal{N}[\eta] &= \mathcal{L}\left(\frac{1}{2}\eta_t^2+\frac{1}{2}\mathcal{H}\eta_t^2-\eta\Omega^2\eta\right)-(\eta_t\mathcal{H}\eta_t+\eta S\eta_x)_x,\\
\mathcal{C}[\eta] &= \left(\eta\eta_t\eta_{xt}-\frac{1}{2}\eta^2\mathcal{L}S\eta_x+\eta_t\mathcal{H}(\eta\mathcal{H}\eta_t)_x-\eta\left(\frac{1}{2}\eta_t^2+\frac{1}{2}\mathcal{H}\eta_t^2-\eta\Omega^2\eta\right)_x\right)_x\\
&\quad+\mathcal{L}\left(\eta_t\mathcal{L}\mathcal{H}(\eta\mathcal{H}\eta_t)-\mathcal{H}\eta_t\mathcal{L}(\eta\mathcal{H}\eta_t)-\eta\mathcal{L}\left(\frac{1}{2}\eta_t^2+\frac{1}{2}\mathcal{H}\eta_t^2-\eta\Omega^2\eta\right)\right)\\
&\quad+\mathcal{L}\left(\frac{1}{2}(\eta_x)_x^3+\eta(\mathcal{H}\eta_t\mathcal{L}\mathcal{H}\eta_t+\eta_t\mathcal{L}\eta_t)+\frac{1}{2}\eta^2S\eta_{xx}\right).
\end{aligned} \tag{2.6}$$

The operators are defined as $\mathcal{L}=(-\partial_x^2)^{1/2}$, $S=(1-\partial_x^2)$, $\Omega^2=S\mathcal{L}$ and $\mathcal{H}$ is the Hilbert transform, whose Fourier symbol is $-i\mathrm{sign}(k)$.

3. Forcing and solitary waves

In this section we consider the forced problem ($\delta\neq 0$) for moving pressure distributions $P:=P(x-ct)$. Solutions of Eq. (2.5), with a prescribed $P(x,t)$, as for example in (3.9), depends on δ and c. The constant ϵ is a measure of the size of η. To understand the behavior of small solutions, we consider the expansion

$$\eta=\eta_1+\epsilon\eta_2+\epsilon^2\eta_3+\dots. \tag{3.1}$$

The leading order solution η_1 solves the linear problem

$$\eta_{1,tt}+(1-\partial_x^2)(-\partial_x^2)\eta_1=\delta\mathcal{L}P,$$

with solution

$$\hat{\eta}_1(k,t)=\frac{\delta|k|\hat{P}(k)}{(1+k^2)|k|-c^2k^2}e^{ickt}=\frac{\delta|k|\hat{P}(k)}{\Lambda(k,c)}e^{ickt}, \tag{3.2}$$

where $\hat{P}(k)$ refers to the Fourier transform of the function $P(x)$ in the dual variable k. Such expansions have been applied to the potential flow equations since Stokes [45], for whom these expansions are now named [14]. When the denominator in Eq. (3.2) is bounded away from zero (the forcing is subcritical and not near-resonant) the nonlinear solution can be expressed as a regular perturbation expansion whose leading term is (3.2). The denominator $\Lambda(k,c)$ is nonzero for subcritical speeds, c below the phase speed minimum of Fig. 1. In this work we consider the special case, where $0<\Lambda(k_0,c)\ll 1$. Although $\Lambda(k_0,c)$ may be small at any value of k_0, we focus on the case $k_0=1$ – the phase speed minimum. Implicit in the perturbation expansion (3.1) is the assumption that $\eta_j=O(1)$; thus from the leading order solution we require that $\delta=O(\Lambda(k,c))$ (see Fig. 2).

To further understand the forced solution, we consider a pressure distribution supported at a single wavenumber $P(x)=pe^{ik_0(x-ct)}+*$; here and throughout $*$ refers to the complex conjugate. We will focus on the case $\delta\approx\Lambda(k_0,c)$. The perturbation solution is

$$\begin{aligned}
\eta_1 &= \frac{\delta|k_0|p}{\Lambda(k_0,c)}e^{ik_0(x-ct)}+*,\\
\eta_2 &= \frac{-\delta^2N(k_0,k_0)|k_0|^2p^2}{2\Lambda(2k_0,c)\Lambda(k_0,c)^2}e^{2ik_0(x-ct)}+*,\\
\eta_3 &= \frac{\delta^3|k_0|^3N(k_0,k_0)N(2k_0,k_0)p^3}{2\Lambda(3k_0,c)\Lambda(2k_0,c)\Lambda(k_0,c)^3}e^{3ik_0(x-ct)}+\frac{\delta^3|k_0|^3N(k_0,k_0)N(2k_0,-k_0)|p|^2p}{2\Lambda(2k_0,c)\Lambda(k_0,c)^4}e^{ik_0(x-ct)}\\
&\quad-\frac{\delta^3|k_0|^3C(k_0,k_0,k_0)p^3}{3\Lambda(3k_0,c)\Lambda(k_0,c)^3}e^{3ik_0(x-ct)}-\frac{\delta^3|k_0|^3C(k_0,k_0,-k_0)|p|^2p}{\Lambda(k_0,c)^4}e^{ik_0(x-ct)}+*.
\end{aligned}$$

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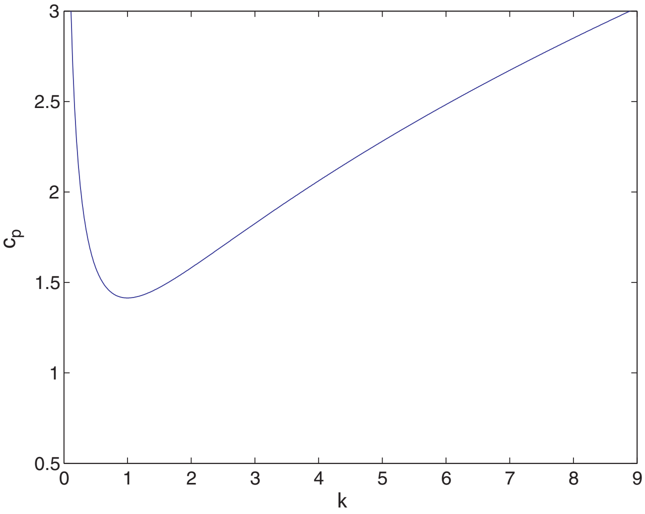

Fig. 1. The phase speed $c_p = \omega/k$ for Eq. (2.5) has a minimum at $k = 1$.

The coefficients $N(k_1, k_2)$ and $C(k_1, k_2, k_3)$ of the harmonics come from the evaluation of the quadratic terms in the nonlinearity at wavenumbers k_1 and k_2, and the cubic terms in the nonlinearity at k_1, k_2 and k_3 respectively. Explicit formulas for these coefficients may be found in Appendix A.

We consider the limit $\Lambda(k_0, c) \to 0$ with $\delta\Lambda(k_0, c)^{-1}$ fixed. In this case, the third order correction violates the perturbation assumption – as will higher order terms proportionate to $e^{ik_0(x-ct)}$. The problem results from a quartet interaction, $k_0 = k_0 + k_0 - k_0$. We propose that emission of localized wave-packets can balance this term. To see this, consider the expansion

$$\eta = \eta_1 + \epsilon\eta_2 + \epsilon^2\eta_3 + A(\epsilon(x - ct), \epsilon^2 t)e^{ik_0x - i\omega t} + * + \dots . \tag{3.3}$$

where k_0 is the carrier wave frequency and ω is the linear dispersion relation evaluated at k_0. As with the previous expansion, terms at $O(1)$ and $O(\epsilon)$ are nonsecular. The $O(\epsilon^2)$ terms yield a forced equation for η_3

$$\begin{aligned}\eta_{3,tt} &+ \Omega^2\eta_3 + \chi\left((|p|^2 + |A|^2)pe^{-ick_0t} + A^2\bar{p}e^{-2i\omega t + ick_0t} + p^2\bar{A}e^{-2ick_0t + i\omega t}\right)e^{ik_0x} \\ &+ (-2i\omega A_\tau + \lambda A_{\xi\xi} + \chi(|A|^2 + |p|^2)A)e^{ik_0x - i\omega t} + r.p.t. + * = 0,\end{aligned} \tag{3.4}$$

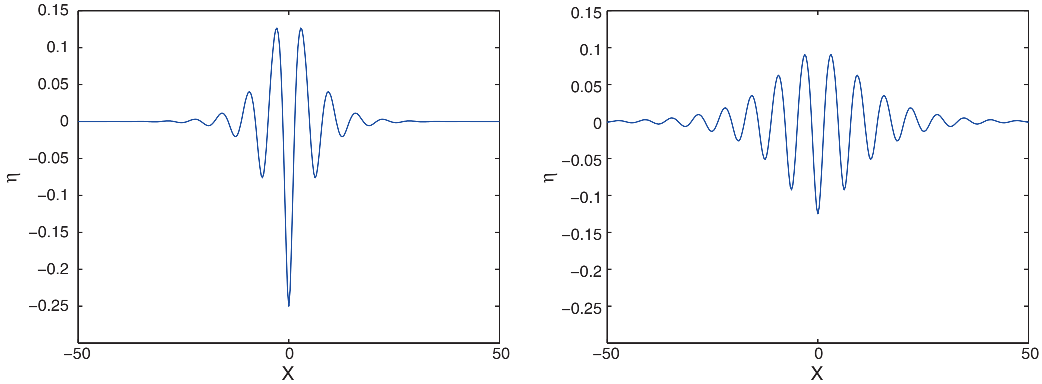

Fig. 2. Examples of numerically computed solitary wave solutions to Eq. (2.5) with $\delta = 0$ with $\|\eta\|_\infty = 0.25$ (left) and $\|\eta\|_\infty = 0.125$ (right). Only a portion of the computational domain is shown.

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where $\xi=\epsilon(x-ct)$ and $\tau=\epsilon^2 t$. Here r.p.t. stands for terms which are part of a regular perturbation expansion, for example proportionate to $e^{3ik_0(x-ct)}$. We consider the case where the forcing is near resonant, with the scaling

$$\Lambda(k_0, c) = \omega^2 - k_0^2 c^2 = \epsilon^2,$$

which, combined with the assumption that $\delta\Lambda(k_0, c)^{-1}$ is fixed, implies both that $\delta = O(\epsilon^2)$ and the scaling

$$c = c_p - \frac{1}{2|c_p|k_0^2}\epsilon^2 + O(\epsilon^4).$$

Substituting this expansion for the speed into Eq. (3.4), and defining $\beta = pe^{i\tau/(2\omega)}$ the term which forces η_3 resonantly is

$$(-2i\omega A_\tau + \lambda A_{\xi\xi} + \chi\left\{(|A|^2 + |\beta|^2)A + (|A|^2 + |\beta|^2)\beta + A^2\bar{\beta} + \beta^2\bar{A}\right\})e^{ik_0x - i\omega t}.$$

Thus if A is governed by the forced NLS equation

$$-2i\omega A_\tau + \lambda A_{\xi\xi} + \chi\left\{(|A|^2 + |\beta|^2)A + (|A|^2 + |\beta|^2)\beta + A^2\bar{\beta} + \beta^2\bar{A}\right\} = 0, \tag{3.5}$$

then the correction η_3 is bounded independent of ϵ. This forced nonlinear Schrödinger Eq. (3.5), NLS, can be thought of as a solvability condition for the ansatz (3.3). NLS is a generic model, arising in a number of situations [13,39], both with and without forcing [10,34]. To our knowledge, Eq. (3.5) is the first instance of this combination of forcing and nonlinearity derived from the water wave problem. For Eq. (2.5), as well as the deep water potential flow equations, the coefficients of Eq. (3.5) for $k_0=1$ are $\omega = \sqrt{2}$, $\lambda = -1/2$ and $\chi = -11/2$. The NLS coefficients for potential flow were first computed in [37], then later independently in [16,22]. The relative signs of λ and χ are predictive in the dynamics of solutions of the Nonlinear Schrödinger Eq. (3.5). At $k_0=1$, the NLS for Eq. (2.5), as well as that of potential flow, is of the focusing type, $\lambda\chi > 0$. Thus Eq. (3.5) has solitary wave solutions

$$A(\xi, \tau) = \left|\frac{2\lambda}{\chi}\right|^{1/2} \operatorname{sech}(\xi) e^{-i\lambda/(2\omega)\tau}, \tag{3.6}$$

when $\beta = 0$ [13]. Thus Eq. (3.5) also has solitary waves solutions

$$A = \beta(\xi, \tau) = \left|\frac{2\lambda}{6\chi}\right|^{1/2} \operatorname{sech}(\xi) e^{-i\lambda/(2\omega)\tau}. \tag{3.7}$$

This solution inspires the forcing of Eq. (2.5) with a near monochromatic pressure distribution, with a hyperbolic secant as the envelope.

The solitary waves solutions of NLS correspond to envelopes of wavepackets in Eq. (2.5). Near the phase speed minimum, $k_0 \approx 1$, these wavepackets are approximate traveling waves. The group velocity $c_g = \omega'(k)$ is equal to the phase speed $c_p = \omega/k$ as

$$c_{p'}(k) = \left(\frac{\omega}{k}\right)' = \frac{1}{k}(c_g - c_p) = 0.$$

This correspondence has been noted previously in [1,4,32]. Although the NLS Eq. (3.5) predicts the existence of wavepacket solitary waves, NLS fails as a model for solitary wave dynamics [2,3]. To to study solitary wave dynamics, we use instead Eq. (2.5).

A rescaled version of Eq. (2.5) was numerically simulated,

$$\eta_{tt} + \Omega^2\eta + Q[\eta] + C[\eta] = \epsilon\delta\mathcal{L}P, \tag{3.8}$$

with a traveling, locally-supported, near monochromatic ($k_0 \approx 1$) pressure distribution

$$P(x, t) = 2\operatorname{sech}(\epsilon(x - ct))\sin(x - ct). \tag{3.9}$$

A quiescent free surface, $\eta = 0$, is used as initial data for all simulations. The characteristic plane and wave profiles for $\epsilon = 0.1$ are in Figs. 3 and 4, respectively. Depression solitary waves downstream are generated downstream of the traveling pressure forcing. The argument connecting NLS and Eq. (2.5) predicts solitary waves for arbitrary ϵ. We

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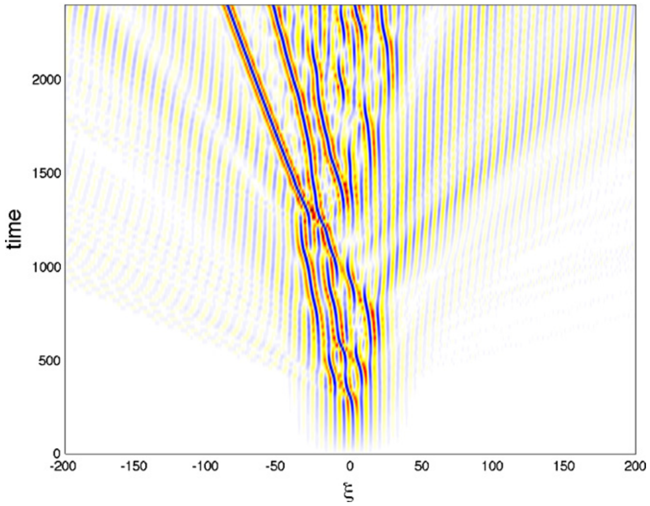

Fig. 3. A forced solution in the *xt*-plane, with $\delta=\epsilon^2=0.01$, moving at $c=\sqrt{2}-\epsilon^2/\sqrt{8}$. The frame is moving with the localized forcing (at $\xi=x-ct$). Solitary waves are generated downstream (to the left) of the forcing. Only a portion of the computational domain is shown.

observe similar results for $\epsilon\approx 0.1$. Both the time and spatial scales of the NLS prediction are proportionate to inverse powers of ϵ. This scaling makes the simulation of significantly smaller ϵ numerically impractical.

The numerical method used to generate Figs. 3 and 4 uses Fourier collocation for spatial derivatives. The time evolution is then conducted using integrating factors and 4th-order Runge–Kutta, as in [30]. The simulations use a domain size of 1024 space units, with 8192 Fourier modes, thus a spacial step of $\Delta x=0.125$, and a time step of $\Delta t=0.002$. The absorbing boundary condition of [20] is implemented on the boundary, although the character of solutions is not sensitive to the use of this boundary condition.

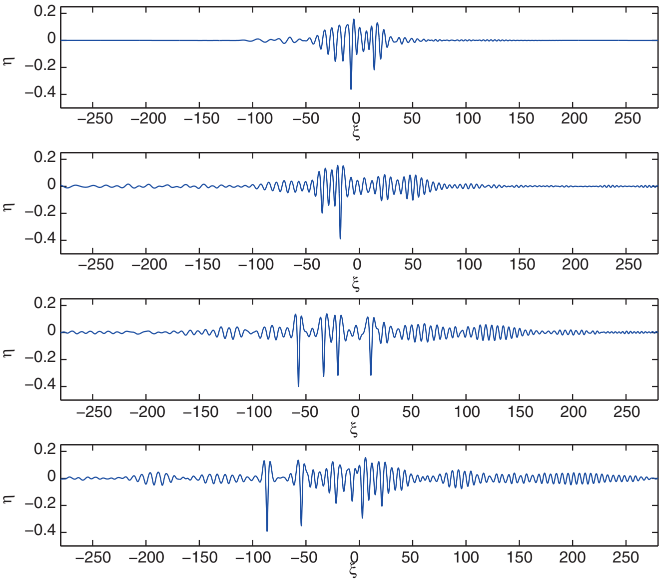

Fig. 4. Profiles of a forced solution, with $\delta=\epsilon^2=0.01$, moving at $c=\sqrt{2}-\epsilon^2/\sqrt{8}$. Snapshots are at $T=600$, 1200, 1800, and 2400. The frame is moving with the localized forcing (at $\xi=x-ct=0$). Two solitary waves (the two largest troughs on the left portion of the last panel) are generated downstream (to the left) of the forcing. Only a portion of the computational domain is shown.

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4. Conclusion

A weakly nonlinear model equation for deep water capillary-gravity waves is derived. This model approximates the potential flow equations to cubic order and includes the effect of surface pressure distributions. Asymptotic solutions for subcritical, near-resonant, traveling pressure distributions are explored using a Stokes-like expansion in the wave amplitude. Quartet interactions are shown to violate the perturbation hypothesis unless a slowly varying wave packet is included in the leading order solution. The envelope of this wavepacket is governed by a forced cubic nonlinear Schrödinger equation (NLS). For a particular forcing, the forced NLS equation has solitary wave solutions. Solitary wave solutions of the forced NLS correspond to approximate wavepacket solitary wave solutions of the model equation.

A cubic approximation of the potential flow equations, written in terms of the free surface displacement, is numerically simulated. Wavepacket solitary waves are observed downstream of a localized, near monochromatic forcing. Solutions of the approximate model are presented on time and amplitude scales relevant to the potential flow equations. In order better understand the physical experiments, viscous effects should be included. This is an area of ongoing research.

Appendix A. Interaction coefficients

When two harmonics of frequencies k_1 and k_2 interact in the nonlinear terms of Eq. (2.5), they generate harmonics of other frequencies. To find the amplitude of the generated harmonics one must evaluate the nonlinearity at these frequencies. In Section 3, we compute the interaction coefficients in a model equation of the form

$$\eta_{tt} + \mathcal{L}\eta + \mathcal{N}(\eta, \eta) + \mathcal{C}(\eta, \eta, \eta) = 0.$$

When $\eta = a_1 e^{ik_1 x} + a_2 e^{ik_2 x}$, the quadratic term of the nonlinearity includes a contribution

$$\mathcal{N}(\eta, \eta) = a_1 a_2 N(k_1, k_2) e^{i(k_1+k_2)x}$$

which for Eq. (2.5) is defined

$$\begin{aligned} N(k_1, k_2) = {} & |k_1 + k_2|(c^2(|k_1k_2| - k_1k_2) - |k_1|(1 + k_1^2) - |k_2|(1 + k_2^2)) + k_1k_2(2 + k_1^2 + k_2^2) \\ & + k_1^2(1 + k_1^2) + k_2^2(1 + k_2^2) + c^2 k_1 k_2 \left(|k_1| + |k_2| + \frac{k_1k_2}{|k_1|} + \frac{k_1k_2}{|k_2|} \right). \end{aligned}$$

Similarly, if $\eta = a_1 e^{ik_1 x} + a_2 e^{ik_2 x} + a_3 e^{ik_3 x}$, the cubic term of the nonlinearity will include a term

$$\mathcal{C}(\eta, \eta, \eta) = a_1 a_2 a_3 C(k_1, k_2) e^{i(k_1+k_2+k_3)x}$$

which for Eq. (2.5) is

$$\begin{aligned} C(k_1, k_2, k_3) = {} & (k_1 + k_2 + k_3)\{c^2(k_1^2(k_2 + k_3) + k_2^2(k_1 + k_3) + k_3^2(k_1 + k_2)) + \frac{1}{2}k_1|k_1|(1 + k_1^2) \\ & + \frac{1}{2}k_2|k_2|(1 + k_2^2) + \frac{1}{2}k_3|k_3|(1 + k_3^2) - c^2k_1|k_2 + k_3|(|k_2| + |k_3|) - c^2k_2|k_1 \\ & + k_3|(|k_1| + |k_3|) - c^2k_3|k_2 + k_1|(|k_2| + |k_1|) - (k_1 + k_2)(c^2(k_1k_2 - |k_1k_2|) \\ & + |k_1|(1 + k_1^2) + |k_2|(1 + k_2^2)) - (k_1 + k_3)(c^2(k_1k_3 - |k_1k_3|) + |k_1|(1 + k_1^2) \\ & + |k_3|(1 + k_3^2)) - (k_2 + k_3)(c^2(k_3k_2 - |k_3k_2|) + |k_3|(1 + k_3^2) + |k_2|(1 + k_2^2))\} + |k_1 \\ & + k_2 + k_3|\{c^2k_1(k_2 + k_3)(|k_2| + |k_3|) + c^2k_2(k_1 + k_3)(|k_1| + |k_3|) + c^2k_3(k_2 + k_1)(|k_2| + |k_1|) \\ & - c^2|k_1||k_2 + k_3|(|k_2| + |k_3|) - c^2|k_2||k_1 + k_3|(|k_1| + |k_3|) - c^2|k_3||k_2 + k_1|(|k_2| + |k_1|) \\ & + |k_1 + k_2|(|k_1|(1 + k_1^2) + |k_2|(1 + k_2^2)) + |k_2 + k_3|(|k_2|(1 + k_2^2) + |k_3|(1 + k_3^2)) \\ & + |k_1 + k_3|(|k_1|(1 + k_1^2) + |k_3|(1 + k_3^2)) + c^2|k_2 + k_3|(k_2k_3 - |k_2k_3|) + c^2|k_1 \end{aligned}$$

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$$
\begin{aligned}
&+k_3|(k_1k_3-|k_1k_3|)+c^2|k_2+k_1|(k_2k_1-|k_2k_1|)+\frac{3}{2}(k_1+k_2+k_3)k_1k_2k_3+c^2k_1(k_1|k_2|\\
&-k_2|k_1|)+c^2k_2(k_2|k_1|-k_1|k_2|)+c^2k_1(k_1|k_3|-k_3|k_1|)+c^2k_3(k_3|k_2|-k_2|k_3|)\\
&+c^2k_2(k_2|k_3|-k_3|k_2|)+c^2k_3(k_3|k_1|-k_1|k_3|)-(1+k_1^2)k_1^2-(1+k_2^2)k_2^2-(1+k_3^2)k_3^2\}.
\end{aligned}
$$

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