Conservation Laws

A fundamental tool in deriving the equations of applied math and physics is the idea that some quantities can be tracked in a physical system. This idea is used to create some balance laws for the system, which in turn is used to write an equation. Common examples include conservation of mass, conservation of energy, and momentum balance. (The third is often called conservation of momentum, which is a bit of a misnomer, as the amount of momentum is rarely conserved, but instead changes in momentum are balanced by some force exerted on a system.) Following are two examples to illustrate how to use the idea of balance laws to write down partial differential equations.

0.1 Heat Equation

Say we have a stationary, one dimensional, infinite rod, and we measure position along the rod x. Now we want to describe the energy E in some region, say from x_0 to x_1. If we make the assumption that the rod is not under any mechanical stress and is not moving, then the energy is proportional to the temperature of the rod, so $E = \gamma T$ where γ is a constant and T to be temperature. Now if we want to write a balance law for the enery between x_0 and x_1 we write

$$(\int_{x_0}^{x_1} \gamma T)_t = \text{change in total energy between } x_0 \text{ and } x_1$$

and we should set this equal to the things that cause the total energy to change. These things are typically energy **flux** through the boundaries (motion of energy in and out), and energy production/destruction in the interior. The flux in this case we will write $F(T)$, and take it to be the energy moving to the right. This allows us to write a balance equation

$$(\int_{x_0}^{x_1} \gamma T)_t = \gamma F(T(x_0)) - \gamma F(T(x_1)) + \int_{x_0}^{x_1} B(x,t)$$

Now a common problem for conservation law derivations is how does one find the flux, F. For temperature we are lucky and Fourier has done it for us (Here it is somewhat intuitive that the heat flow between two regions should be proportionate to the difference in their respective temperatures)

$$F(T) = -k(x)\nabla T$$

Where here $k(x)$ is the thermal conductivity of the medium. Now we can use the fundamental theorem of calculus to rewrite the left hand side of the balance law to get

$$(\int_{x_0}^{x_1} \gamma T)_t = -\int_{x_0}^{x_1} \gamma F(T)_x + \int_{x_0}^{x_1} B(x,t)$$

If we now take the time derivative inside the integral sign and replace the flux with Fourier's definition, we get

$$\int_{x_0}^{x_1} \gamma T_t - \gamma(k(x)T_x)_x - B(x,t) = 0$$

Now we notice that x_0, and x_1 were arbitrary. If the integral of a continuous function is zero regardless of the endpoints, then we know that the function is zero so we have that the argument of the integral is zero pointwise. If we presume that $k = \textit{constant}$, and $B = 0$ we have the most familiar version of the heat equation

$$T_t = k_0 T_{xx}$$

0.2 Traffic Flow

A common toy problem in conservation laws is a conservation law for cars on a highway. Denote $\rho(x,t)$ to be the number of cars at a point x and time t. We can follow a similar procedure as the heat equation, looking at the change of the density of cars in a region

$$\partial_t \left(\int \rho \right) = \text{Cars in} - \text{Cars out}$$

To see what the flux should be here we can just examine the units, the left hand side has units cars/time, so if we introduce u=velocity of the cars, with units length/time, then we can write a flux $F = \rho u$. As before this leads to a pde

$$\rho_t + (\rho u)_x = 0$$

This is the same equation that arises in fluid flow problems, and is generally called conservation of mass. Regardless of the situation we need another equation to close this system. For traffic flow the tradition is to write $u = u_{max}/\rho_{max}(\rho_{max} - \rho)$. This style of speed dependence on density fits with our intuition, low density will lead to large speed, high density to low speed. With the appropriate change of variables, we can write a single equation for ρ

$$\rho_t + (1/2\rho^2)_x = 0$$

This equation is known as the Inviscid Burgers' Equation.

0.3 Further Problems

- Conservation of Mass, Traffic flow / burgers' equation
- Bacteria and Attractant, see qual Jan 2004.
- Momentum Balance, Navier Stokes eqn.

0.4 References

- Barenblatt - The last chapter does the porous medium eq. with darcy's law
- Haberman - Does a good job with the heat equation.
- Leveque - Trafffic flow and gas dynamics.