Generalized Functions

In a previous lecture we derived a special self similar solution to the heat equation $u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{x^2/(4t)}$. We noted here that $\int_R u(x,t) = 1$ and $\lim_{t\to 0} u(x,t) = 0$ for $x \neq 0$, and $\lim_{t\to 0} u(x,t) = \infty$ for $x = 0$. This limit is called the delta function.

$$\lim_{t\to 0} \frac{1}{\sqrt{4\pi t}} e^{(x-x_0)^2/(4t)} = \delta_{x_0}(x)$$

The delta function has some important properties that makes it useful for solving linear pde's.

- Even: $\delta(x) = \delta(-x)$
- Normalized: $\int_R \delta(x)dx = 1$
- Simple convolution: $\int_R f(x)\delta(x-x_0)dx = f(x+0)$

We can verify this convolution property with stationary phase.

We can also define the derivative of the delta function, $\delta'(x)$, which is odd, has zero mean, and when convolved with $f(x)$ gives $\delta' * f = -f'(x)$.

1 Fourier Transforms of Generalized functions

It is not hard to calculate the fourier transform of a delta function using the above properties. In the previous lecture on fourier transform we ignored the constant factor of $\frac{1}{2\pi}$, here we will track it.

$$\mathcal{FT}(\delta) = \frac{1}{2\pi} \int \delta(x) e^{-ikx} dx = \frac{1}{2\pi}$$

Notice that if we try to take the inverse fourier transform the integral does not exist in the traditional sense, however we still say that

$$\mathcal{IFT}(\frac{1}{2\pi}) = \delta(x)$$

1.1 Important Fourier Transforms

There are some important fourier transforms that come associated with generalized functions.

$f(x)$	$\hat{f}(k)$
$\delta(x)$	$\frac{1}{2\pi}$
$\delta'(x)$	$\frac{ik}{2\pi}$
$f(x-x_0)$	$e^{ikx_0}\hat{f}(k)$
x	$i\delta'(k)$

1.2 Example: Heat Equation

We can now solve the heat equation with a delta function as initial data.

$$u_t = u_{xx}$$
$$u(x,0) = \delta(x)$$

Taking the fourier transform gives $u = \frac{1}{2\pi} \int e^{-k^2 t} e^{ikx} dk$, which we have already evaluated, (complete the square in the exponent and do a change of variables) and gives exactly the self similar solution.

1.3 Solving ODE's with generalized functions

Delta functions can be used to find a larger class of solutions to ODE's, by noting

$$\int x\delta'(x) = -\int \delta(x)$$

This means that in the sense of distributions

$$x\delta'(x) + \delta(x) = 0$$

Here the equality in the sense of distributions means that it holds when we integrate against test functions like

$$\int x\phi(x)\delta'(x) = -\int \phi(x)\delta(x)$$

This knowledge is useful for solving the ode's, especially those we get from solving linear pde's with the fourier transform. For instance $(xf)' = 0$ has solution, $f = c/x$, but also $f = c/x + d\delta(x)$.

1.4 Example: Helmholtz Equation

Solve the Helmholtz equation in 1-D

$$u_{xx} + \lambda^2 u = 0$$

Now we know in advance that the solutions are $\sin(\lambda x)\cos(\lambda x)$, however now we can justify this with the fourier transform. Taking the fourier transform of this equation gives

$$(\lambda^2 - k^2)\hat{u} = 0$$

Which says that $\hat{u} = 0$, or $(\lambda^2 - k^2) = 0$. But this is another way of saying that the solution is $\hat{u} = C\delta(\lambda^2 - k^2)$. Now if we use the definition of the fourier transform, we get

$$u = \int C\delta(\lambda^2 - k^2)e^{ikx}dk = Ce^{i\lambda x} + Ce^{-i\lambda x}$$

so we recover the $\sin(\lambda x)$ and $\cos(\lambda x)$ as expected.

Qual : Aug 05 problem 4, Jan 05 problem 6, Aug 04 problem 1, Aug 03 problem 6

References: Zauderer