1 Differential Operators

Consider the problem

$$Lu = f \qquad \text{with} \qquad Bu = g$$

Where L is a differential operator, and B corresponds to some boundary conditions. The problem we pose here is when does a unique solution exist. To understand the answer we begin with some definitions. An operator is **Self adjoint** if $< Lu, v >=< u, Lv >$ for all functions u, v, (we do sometimes say self adjoint over a space of functions rather than all functions). Some operators are not self adjoint, thus we define the adjoint of an operator as L^* which satisfies $< Lu, v >=< u, L^* v >$.

Theorem: Friedholm Alternative

- The solution to Lu=f is unique iff $Lu = 0$ has only the trivial solution u=0.
- The solution to Lu=f exists iff $< f, v >= 0$ for all v such that $L^* v = 0$.

We can understand the first statement in this theorem via linear algebra. If L is a matrix this is a well known fact. The second statement we can see if we integrate against a test function and then use the adjoint of L

$$< f, v >=< Lu, v >=< u, L^* v >$$

So if $L^* v = 0$, then $< f, v >= 0$. For the other direction, we can construct solutions by projecting u onto eigenfunctions.

1.1 Example

Take $L = \partial_x^2 + 1$, $f = \sin(x)$ and homogeneous boundary conditions at $x = \pm\pi$.

$$u_{xx} + u = \sin(x)$$
$$u(-\pi) = u(\pi) = 0$$

If we look in the context of this theorem, we see that there should be no solution to this problem. Indeed we know the solution without the boundary conditions to be $u(x) = A\sin(x) + B\cos(x) - x/2\cos(x)$, where A and B depend on the boundary conditions. But if we impose the boundary conditions we see that there can be no solution of this form. Notice here that $< sin(x), sin(x) >= \pi \neq 0$, so the second part of the Friedholm alternative guarantees no solution exists.

2 Sturm Liouville Problems

A common second order self adjoint problem is

$$(p(x)u_x)_x + q(x)u = f$$

This problem with homogeneous or periodic boundary conditions is known as a Sturm-Liouville problem. If p,q are continuous there is an associated eigenvalue problem,

$$(p(x)u_x)_x + q(x)u + \lambda u = 0$$

There are a number of important theorems associated with this type of problem

- All the eigenvalues are real
- There is an infinite number of eigenvalues, with a smallest λ_1 and no largest.
- For each eigenvalue there is an eigenfunction.
- The eigenfunctions form a complete basis for C^1.
- The eigenfunctions are orthogonal.
- Eigenfunctions are related to eigenvalues by the Rayleigh Quotient.

Exercise: Show that the eigenfunctions are orthogonal.

2.1 Example : Constructing Solutions

When we given a Sturm-Liouville problem, because we know that there is a complete basis for C^1 as solutions to the eigenvalue problem, we can solve $Lu = f$, by decomposing f into a sum of the basis functions $f = \sum a_n \phi_n$, where ϕ_n come from $Lu = \lambda u$. Then we can trivially find that $u = \sum a_n/\lambda_n \phi_n$. To find the a_n we can use the same procedure as if we were finding the coefficients of a fourier series.

3 Rayleigh Quotient

Another important topic here is the idea of the Rayleigh quotient. By taking the inner product of an eigenfunction with itself we can solve for the eigenvalue via

$$\lambda = \left(p\phi\phi_x|_a^b + \int_a^b p(\phi_x)^2 - q\phi^2 \right) / (\int_a^b \phi^2)$$

Because we know that there is a minimum eigenvalue, we can use this to write a minimization principle. Moreover, since the eigenfunctions form a basis for the continuous functions, we don't need to know the eigenfunctions to find upper bounds on this minimum.

3.1 Example: Rayleigh Quotient

Write a minimization principle for the smallest eigenvalue to

$$\phi_{xx} + x^2\phi = \lambda\phi$$
$$\phi(0) = \phi(1) = 0$$

And calculate explicitly an upper bound for this eigenvalue. Here we simply write the Rayleigh quotient.

$$R(\phi) = (\int_0^1 (\phi_x)^2 - x^2\phi^2 dx)/(\int_0^1 \phi^2 dx)$$

Notice here that without loss of generality we can assume that $|\phi|_2 = 1$, else we can renormalize, so we really just need to look for the minimum value of

$$R_2(\phi) = \int_0^1 (\phi_x)^2 - x^2\phi^2 dx$$

Over all functions with $\phi(0) = \phi(1) = 0$. Now any function that satisfies these boundary conditions and has L_2 norm equal to 1 will provide an upper bound. For simplicity we will choose one what makes the integral easy to evaluate, ie $\phi = x(x-1)\sqrt{30}$ which gives, $\lambda_1 < 68/7$.

Exercise: How might we improve on this bound?

References: Keener, Strang, Haberman
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