

# 1 Well Posedness

Def: A pde is said to be **well posed** if it has continuous dependence on initial data. Conversely an ill posed pde has initial, or boundary data, where an infinitesimal perturbation can grow unbounded away from the unperturbed solution. Notice that if we can prove that the solution is uniformly bounded everywhere then it is well posed, but on the other hand we may also have unbounded solutions which are not ill posed.

To show a pde is ill-posed, all we have to do is find a perturbation that grows unboundedly away from another solution, independent of the size of perturbation. In other words, we cannot construct a bound for the difference in solutions based on the size of the difference in initial data. For constant coefficient PDEs, an easy way to do this is to look at fourier type solutions. Then if the growth rate of the solution grows unbounded as a function of frequency, we can find such a perturbation.

## 1.1 Example: Backward Heat Equation

This is perhaps the most common example of an ill-posed pde.

$$\begin{aligned}u_t &= -u_{xx} \\ u(0, x) &= f(x)\end{aligned}$$

To see if this is well or ill posed, we look at a simple case for the initial data,  $f(x) = e^{ikx} + c.c.$ . Then we can easily find the solution,  $u(x, t) = e^{ikx - i\omega t}$ , with dispersion relation  $i\omega = k^2$ , so the solution is  $u(x, t) = e^{ikx + k^2 t}$ . Now we see that the growth rate of solutions depends on the frequency of the initial data. Thus if we consider  $f(x) = e^{ix} + \epsilon e^{2ix}$ , for long time this will grow like  $e^{4t}$ , where the unperturbed solution will grow like  $e^t$ . Thus we say that the PDE is ill posed, it does not depend continuously on initial data.

## 1.2 Example: A Boundary Value Problem

Consider the heat equation on the half plane,  $t > 0, x > 0$ .

$$\begin{aligned}u_t &= u_{xx} \\ u_x(0, t) &= h(t) \\ u(0, t) &= g(t)\end{aligned}$$

Again we will look at a simple case,  $g(t) = e^{i\omega_0 t}$ ,  $h(t) = 0$ . Again we will look for solutions  $e^{ikx - i\omega t}$  but here we see that  $\omega$  is given. Thus we must solve for  $k = k(\omega_0)$ . If we solve this problem we get

$$u(x, t) = 1/2 \left( e^{\sqrt{\omega_0}/\sqrt{2}(i+1)x} + e^{-\sqrt{\omega_0}/\sqrt{2}(i+1)x} \right) e^{i\omega_0 t}$$

Here again we can find a perturbation to show that this is an ill posed pde. We just choose  $u(0, t) = e^{i\omega_0 t} + \epsilon e^{i\Omega t}$  with  $\Omega \gg \omega_0$ .

## 1.3 Homework: Laplaces Equation

Show that Laplaces equation on the half plane is ill posed if we specify  $u(x, 0) = e^{ikx}$ ,  $u_y(x, 0) = 0$ . How can we change the boundary conditions to make it well posed?

An important point here is that the exponential growth is not what makes the problem ill posed.

## 1.4 Examples: Exponentially Growing, well posed

A simple example of a well posed exponentially growing problem is

$$u_{xx} = a^2 u$$

Here we get two solutions  $u(x) = Ae^{a|x} + Be^{-a|x}$ . Here we can satisfy any two boundary conditions with the proper choice of  $A$  and  $B$ . In this case if we perturb the boundary conditions by a small amount we only perturb

A and B by a small amount, but the exponential growth is still proportionate to  $e^{|a|x}$ .

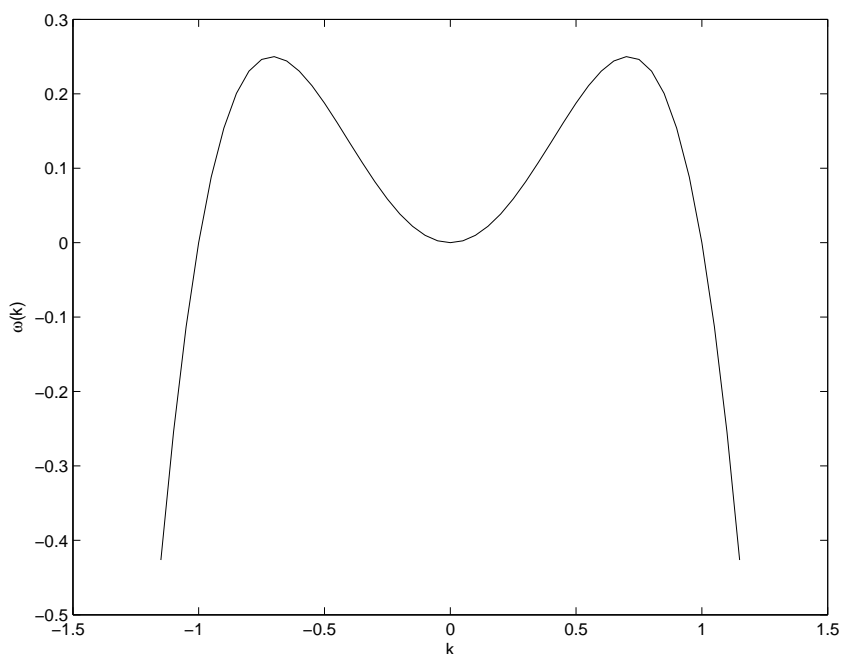
Another example is the PDE

$$u_t = -u_{xx} - u_{xxxx}$$
$$u(0, x) = f(x)$$

Here when we take the fourier transform we get the dispersion relation,  $i\omega = -k^2 + k^4$ . Thus we get solutions

$$u \sim Ae^{ikx - (k^2 - k^4)t}$$

Here there is exponential growth for  $k \in (-1, 1)$ , and exponential decay elsewhere. Also, there is a maximum growth of  $e^{1/4t}$ . Thus this problem is well posed. We cannot perturb  $f(x)$  by  $\epsilon e^{ikx}$  and get arbitrarily large growth away from a solution in finite time.



References: Evans, Haberman  
Qual: Aug 02 problem 6