1 Green's Functions

For linear problems, Green's functions can be a valuable technique for finding solutions. The idea here is to use convolution to switch from the solution to a problem with a delta function as initial data, to the general forced problem.
We do this by writing

$$u(x) = (v * f)(x) = \int_R v(x - x_0) f(x_0) dx_0$$

Which is read the convolution of v with f. Then a common exercise is to show that if v solves the a linear problem with a delta function as initial data then u solves the forced one.

1.1 Example: Heat equation

Show that if v solves

$$v_t = v_{xx}$$
$$v(0, x) = \delta(x)$$

Then $u = v * f$ solves

$$u_t = u_{xx}$$
$$u(x, 0) = f(x)$$

Here we can just plug in definitions, and since convolution is linear, the proof is trivial. One might also ask to find the green's function here v. We have already done this in a previous lecture for the heat equation.
A common question is to find the green's function for a particular problem. Sometimes we may do this by finding the formal fourier solution and evaluating some integrals, sometimes the integrals are intractable. This is where most of the difficult problems arise in the business of green's functions.

1.2 Example: Laplace's Equation with Dirichlet B.C.'s

Find the green's function for the upper half plane

$$u_{xx} + u_{yy} = 0$$
$$u(x, 0) = f(x)$$

Where here $\lim_{y\to\infty} u(x, y) = 0$. Here the procedure is to take the fourier transform in x, yielding

$$\hat{u}_{yy} - k^2 \hat{u} = 0$$

Thus $\hat{u} = Ae^{ikx-|k|y} + Be^{ikx+|k|y}$. Since u decays at infinity, $B = 0$. Also, we can find A in terms of the initial data,

$$A = \hat{f} = \int_R f(t) e^{-ikt} dt$$

So substituting this into u we get

$$u(x, y) = \frac{1}{2\pi} \int_R \int_R f(t) e^{ik(x-t)-|k|y} dk dt$$

Now we can split the domain in k into $k > 0$ and $k < 0$ and get two integrals in k which we can evaluate explicitly.

$$u(x, y) = \frac{1}{2\pi} \int_R \int_{R+} f(t) e^{ik(x-t)-ky} dk dt + \frac{1}{2\pi} \int_R \int_{R-} f(t) e^{ik(x-t)+ky} dk dt$$

or

$$u(x, y) = \frac{1}{2\pi} \int_R \frac{-f(t)}{i(x-t) - y} dt + \frac{1}{2\pi} \int_R \frac{f(t)}{i(x-t) + y} dt$$

Now to find the greens function, recall that $f(t) = \delta(t)$ so these integrals are easy to evaluate, giving

$$g(x, y) = \frac{1}{2\pi} \left(\frac{-1}{i(x) - y} + \frac{1}{i(x) + y} \right) = \frac{y}{\pi(x^2 + y^2)}$$

1.3 Stokes' Rule

In the previous example we saw that there were quite a few steps to determining the greens function to a pde. We would like to avoid this work whenever possible. One trick due to Stokes is to find the greens function for a Neumann problem (one with boundary conditions on the derivative eg $w_y(0,x) = g(x)$) using the greens function for the Dirichlet problem (one with boundary conditions on the function $u(0,x) = f(x)$). The trick here is simple, the greens function to the Neumann Problem is the integral of the solution to the Dirichlet problem.

1.4 Example: Laplace's Equation and Neumann B.C.'s

Find the greens function for

$$v_{xx} + v_{yy} = 0$$
$$v_y(0,x) = f(x)$$

Here we can use g(x,y) from the previous problem, and see that the greens function here will be $h(x,y) = \int_y g(x,y)$.

$$h_{xx} + h_{yy} = \int_y g_{xx} + g_{yy} = 0$$
$$h_y(0,x) = g(0,x) = \delta(x)$$

1.5 The Method of Images

So far all the problems we have done have been homogeneous problems with boundary conditions. We can also write green's functions for problems with a forcing, and on infinite domains nothing really changes. If we add boundary conditions these problems can get more difficult, so we use a technique called the method of images.

1.6 Heat equation on a rod

Find the green's function for

$$u_t = u_{xx}$$
$$u(0,t) = 0$$
$$u(x,0) = f(x), \text{ for } x > 0$$

Here we first find what is called the free space greens function G_f, ie with no boundary conditions over all of R.

$$(G_f)_t = (G_f)_{xx}$$
$$G_f(0,x) = \delta_{x_0}(x)$$

Now we need to add to it another Green's function to satisfy the boundary condition.

$$(G_L)_t = (G_L)_{xx}$$
$$G_L(0,x) = -\delta_{-x_0}(x)$$

The green's function for the problem with boundary conditions is then $G = G_L + G_f$.

1.7 Aside - Polar Coordinates

To do some greens function problems it is helpful to change to polar coordinates. To do this we need to know that

$$\delta(x)\delta(y) = \frac{\delta(r)}{\pi r}$$

and

$$\delta(x)\delta(y)\delta(z) = \frac{\delta(r)}{2\pi r^2}$$

References: Keener, Zauderer, Haberman, Kevorkian

Qual: Jan 05 problem 4, Jan 04 problem 5