1 The Navier Stokes equations

To find equations for a fluid's velocity and density, we will use conservation of mass and momentum balance. Conservation of mass says

$$(\int_\Omega \rho dV)_t = -\int_{\partial\Omega} \rho u \cdot n dA$$

Now we can use Gauss' divergence theorem and the fact that Ω does not depend on t to get

$$\int_\Omega \rho_t + \nabla \cdot (\rho u) = 0$$

or

$$\rho_t + \nabla\rho \cdot u + \rho\nabla \cdot u = \frac{D\rho}{Dt} + \rho\nabla \cdot u = 0$$

Here we introduce $\frac{D\rho}{Dt} = \rho_t + \nabla\rho \cdot u$, the convective derivative, or the derivative moving with the fluid. This notation is motivated if we look at $x(t)$ as moving with the fluid, then the total derivative $\frac{d\rho(x(t),t)}{dt} = \rho_t + \rho_x x_t = \rho_t + u\rho_x = \frac{D\rho}{Dt}$.

Next, look at the momentum balance equation ($F = ma = (mv)_t$), momentum = (mass)(velocity), we get

$$(\int_\Omega \rho u dV)_t = -\int_{\partial\Omega} \rho u \otimes u \cdot n dA + \int_{\partial\Omega} T \cdot n + \int_\Omega F$$

The same steps as before give the equation

$$(\rho u)_t + \nabla \cdot (\rho u \otimes u) = \nabla \cdot T + F$$

So far this is completely general momentum balance. To get the Navier-Stokes equations, we make two approximations. The stress tensor $T_{i,j} = -p\delta_{i,j} + \mu(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j})$, and incompressibility, $\nabla \cdot u = 0$. This gives

$$\rho\frac{Du}{Dt} = -\nabla p + \mu\Delta u + F$$
$$\nabla \cdot u = 0$$

Here we pose the problem as find u and p that satisfy the above. These are the Navier-Stokes equations for an incompressible fluid.

1.1 Reynolds number, Euler, and Stokes

There are some cases where it is convenient to make further approximations, depending on what particular flow regimes we are interested in, ie what Reynold number ($Re = UL/\nu$). To see how we get these approximations we need to nondimensionalize the equations. If we write $X = x/L$, $\tilde{u} = u/U$, $\tau = t/T$ then we will get a new set of equations. Here L=the characteristic length scale, U=the characteristic velocity scale, T=some timescale recovered from L, U and $\nu = \frac{\mu}{\rho}$. Two common choices are the inertial time scale $T = L/U$ or the viscous timescale $T = L^2/\nu$. If we pick the inertial time scale then the nondimensionalized equation is

$$\rho\frac{D\tilde{u}}{Dt} = -\nabla\tilde{p} + \frac{1}{Re}\Delta\tilde{u}$$

Here we imagine that we are looking at $Re >> 1$ so that we can neglect the $\frac{1}{Re}\Delta u$ to get Euler's Equations

$$\rho\frac{Du}{Dt} = -\nabla p$$
$$\nabla \cdot u = 0$$

If we choose the viscous time scale then we get

$$u_t + Re(u \cdot \nabla)\tilde{u} = -\nabla \tilde{p} + \Delta \tilde{u}$$

Here we are usually looking for $Re << 1$ so we neglect the inertia term $Re(u \cdot \nabla)u$ to yield Stokes equations

$$u_t = -\nabla p + \Delta u$$
$$\nabla \cdot u = 0$$

Frequently we are interested in these problems with boundary conditions. For both the Navier Stokes and Stokes equations we impose the condition that the fluid sticks to the boundary, ie the fluid particles on the boundary move at the same speed of the boundary, ie $u \cdot n = U_{wall} \cdot n, u \cdot T = U_{wall} \cdot T$. For the Euler equations we have one less x derivative, so we can impose one less boundary condition. Clearly we do not want fluid moving through walls, so we choose $u \cdot n = U_{wall} \cdot n$. Thus we allow fluid particles to slip along the wall at any tangential velocity in the Euler's equaions, but no tangential slip is is allowed in the Stokes equations. These are the slip and no slip boundary conditions.

1.2 Energy Dissipation

The role of viscousity is to dissipate energy $||u||_2$, much in like the heat equation. To show this in the Navier Stokes equations in 1-D, look at

$$(\int_\Omega 1/2u^2 dx)_t = \int uu_t dx = \int u(-uu_x - p_x + \nu u_{xx})dx = \int 1/3(u^3)_x - pu_x - u_x^2 = \int -u_x^2 < 0$$

To eliminate the pressure term we use incompressibility, the inertial term is a total derivative, and to see the sign of uu_{xx} we integrated by parts. This derivation works only for periodic or homogeneous boundary conditions.

References: Acheson, Johnson

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