Separation of Variables

Given the heat equation,

$$u_t = u_{xx}$$

An obvious question is how to solve it. Now with the proper boundary and initial conditions,

$$u(0, x) = u_0(x)$$
$$u(t, 0) = 0 = u(t, 1)$$

we can solve this equation with separation of variables. The idea is simple. Presume that $u(x, t)$ is the product of two other functions of one variable each, commonly $F(x)$, and $G(t)$. Now plug in $u(x, t) = F(x)G(t)$.

$$G'(t)F(x) = F''(x)G(t)$$

If we presume that F and G are nonzero, then we can divide to get

$$\frac{G'}{G} = \frac{F''}{F}$$

Now the key step. The above equation has a function of t only on the left, and a function of x only on the right. Since this equation is true for all x and t, we know that both sides equal a constant (otherwise we would vary x on one side and keep t fixed, or vice versa, and get a contradiction). Here we will call this constant λ. So we get two ordinary differential equation instead of one partial differential equation

$$G' + \lambda G = 0$$
$$F'' + \lambda F = 0$$

First we solve the first equation and see that $G = A\exp(-\lambda t)$ so if we want out solutions to stay bounded in time, we must add the restriction that $\lambda > 0$. The second equation gives us solutions $F = B\sin(\sqrt{\lambda}x) + C\cos(\sqrt{\lambda}x)$. Imposing the boundary condition at zero gives that $C = 0$. The other boundary condition gives that $\sqrt{\lambda} = n\pi$, where $n \in \mathbb{N}$. Now if $u_0(x) = sin(n\pi x)$ we are done. If not then we notice that the equation we started with was linear, so the sum of two solutions is also a solution. In other words we would be ok if the initial data were a linear combination of $\sin(n\pi x)$. Again we are bailed out by Fourier, and we can use Fourier series to write $u_0(x) = \sum A_n \sin(n\pi x) + B_n \cos(n\pi x)$. This gives us a solution $u(x, t) = \sum_n A_n \sin(n\pi x)e^{-(n\pi)^2 t}$.

Fourier Series

A powerful method for solving linear pde's, the idea is that a large class of functions, defined on and interval $(-L/2, L/2)$ can be expressed as an infinite series of sines and cosines.

$$f(x) = \sum_{n=0}^{\infty} A_n \sin(n\pi/Lx) + B_n \cos(n\pi/Lx)$$

Whether this sum converges depends on the smoothness (differentiablitiy) of f. If it does converge, then we can find the coefficients using the following procedure.

- Multiply the equation by $\sin(M\pi/Lx)$.
- Integrate over the interval.
- Solve for A_M
- Multiply the equation by $\cos(M\pi/Lx)$.
- Integrate over the interval.
- Solve for B_M

Notice this procedure depends heavily on the fact that sin and cos are orthogonal.

Further Problems and References - See Haberman