Stationary Phase

Given

$$F(k) = \int_a^b g(x)e^{ik\phi(x)}dx \tag{1}$$

A question one might ask is how does this integral behave for large k. The simplest way to attack this is to just integrate by parts. If $\phi'(x) \neq 0$ rewrite (1)

$$F(k) = \int_a^b g(x)\frac{ik\phi'(x)}{ik\phi'(x)}e^{ik\phi(x)}dx$$

Recall integration by parts

$$\int udv = vu - \int vdu$$

Then clearly $u = g(x)/ik\phi'(x)$, and $dv = ik\phi'(x)e^{ik\phi(x)}$ will allow us to write

$$F(k) = \frac{g(x)}{ik\phi'(x)}e^{ik\phi(x)}|_a^b - \int_a^b \frac{d}{dx}(\frac{g(x)}{ik\phi'(x)})e^{ik\phi(x)}dx$$

Repeating this procedure we will get a power series for $F(k)$, so long as $\phi^n(x) \neq 0$

$$F(k) = a/k + b/k^2 + ...$$

This is a very simple procedure, but it depends highly on $\phi^n(x) \neq 0$. The procedure for the finding the leading order behavior when $\phi'(x) = 0$ is called stationary phase. We can simplify our problem here by presuming $\phi'(a) = 0$ and $\phi'(x) \neq 0$ elsewhere in (a, b) (if not we will sum over intervals of this form.) Next split the domain into an interval about a and the rest

$$F(k) = \int_a^{a+\epsilon} g(x)e^{ik\phi(x)}dx + \int_{a+\epsilon}^b g(x)e^{ik\phi(x)}dx$$

By the previous statement the second integral is O(1/k). In the first integral we will replace $g(x)$ with $g(a)$, and $\phi(x)$ with $\phi(a) + \phi^n(a)(x-a)^n/n!$, $\phi^n(a)$ is the first nonzero derivative of ϕ at a giving

$$F(k) = \int_a^{a+\epsilon} g(a)e^{ik(\phi(a)+\phi^n(x-a)^n/n!)}dx + O(1/k)$$

Next replace $a+\epsilon$ with ∞ and introduce only another term proportionate to $1/k$. So now we can try to get the leading order behavior from

$$F(k) \sim g(a)e^{ik\phi(a)} \int_0^\infty e^{ik/n!\phi^n(a)s^n}ds$$

Now use the substitution $u = -ik\frac{\phi^n(a)}{n!}s^n$. This changes the integral to

$$F(k) \sim g(a)e^{ik\phi(a)\pm i\pi/2}(\frac{n!}{k|\phi^n(a)|})^{1/n} \int_0^\infty e^{-u}u^{1/n-1}du$$

All that is left is to recognize that $\int_0^\infty e^{-s}s^{z-1} = \Gamma(z)$ and we see that in fact we now have

$$F(k) \sim C/k^{1/n}$$

And we can even explicitly write the constant.

1 Example

Given a fourier integral representation of a solution below, with $W(k)$ monotone in k, find the behavior at long times at a fixed speed, ie where $x = ct$.

$$\phi(x,t) = \int_R F(k)e^{ikx-iW(k)t}dk$$

Replace x with ct in the integral

$$\phi(x,t) = \int_R F(k)e^{i\chi(k)t}dk$$

Now we know that when $\chi'(k) \neq 0$ this integral will decay like $O(1/t)$ (b/c we could integrate by parts). Thus we look at the places where $\chi'(k) = 0$. In this problem this is where $c = W'(k)$ or where c is the phase speed. Now we make the approximations, $F = F(K), \chi = \chi(K) + 1/2(k-K)^2\chi''(K)$.

$$\phi(x,t) \sim F(K)e^{i\chi(K)t}\int_R e^{i/2(k-K)^2\chi''(K)t}dk$$

Now we can evaluate this integral either by a change of variables to a gaussian, or to the gamma function. Either way we get

$$\phi \sim F(K)\sqrt{\frac{2\pi}{t|\chi''(k)|}}e^{i\chi(K)t+i\pi/4\text{sign}(\chi''(K))}$$

Ref: See Bender and Orszag, Whitham
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