Singular Points

Here we will examine linear ode's with nonconstant coefficients.

$$y^n(x) + p_{n-1}(x)y^{n-1} + ...p_1(x)y = 0$$

We will define **ordinary point** as any point where all the coefficients are analytic, and a **singular point** as a point where any of the coefficients are not analytic.
A **regular singular point**, x_0 is one where the coefficients $\{p_0(x)(x-x_0)^n, (x-x_0)p_{n-1}\}$ are analytic in a neighborhood of x_0.

Theorems:

- If there are no singular points, then the solution has a taylor series solution.
- At a regular singular point, the solution may have a Taylor series expansion, or may be of the form $y = (x-x_0)^\alpha A(x)$ where $A(x)$ has a Taylor series expansion whose radius of convergence is at least to the next singular point. (Due to Fuchs, see Bender and Orszag)

To classify the point at infinity, use the map $x = 1/t$, then classify $t = 0$.

1 Method of Dominant Balance

To see the leading order behavior of an irregular singular point, we plug in $y = e^{S(x)}$. This gives an ode for S. Next we presume some terms with in S are small compared to others, until we can solve the ODE. Then we check to see if our assumption was correct near the point we care about. If so, we say we have an asymptotic solution.

1.1 Example

$$x^2y'' + (1+3x)y' + y = 0$$

Here $x = 0$ is an irregular singular point. So try $y = e^{S(x)}$ This gives

$$x^2S''(x) + (S'(x))^2x^2 + (1+3x)S'(x) + 1 = 0$$

Presume that $S''(x)$ and $3xS'$ are small. This gives

$$(S'(x))^2x^2 \sim -S'(x) - 1$$

This gives a quadratic, which we can solve for S'. For small x, these solutions are

$$S' \sim -x^{-2} \text{ and } S' \sim -1$$

Notice here that had we not thrown away $3xS'$ we would have gotten a messier quadratic, but for x small we can still can recover $S'(x) = -x^{-2}$ because we are only looking at x small. The other solution gives $S \sim C$ (for x small), which means y is not changing rapidly, so this solution we should be able to get with Taylor series. (It turns out not in this case, but this is still good logic to stop pursuing this solution.) The next step is to check that the terms we assumed were small are indeed small for this solution. Here they are.
The solution $S' \sim -x^{-2}$ gives $S \sim 1/x$, or $y \sim e^{1/x}$. Now if we want to find the next correction to $S(x)$ we write

$$S(x) \sim 1/x + C(x)$$

Where here we assume $C(x) << 1/x$. The equation for C is then

$$x^2C'' + x^2(C')^2 - (1-3x)C' - 1/x + 1 = 0$$

For $C' << 1/x$, we can approximate this equation by

$$C' \sim -1/x$$

So we get $C(x) \sim \ln(x)$. Next check that if this C is the solution, then the approximated equation is close to the actual equation. This gives $y \sim C1/xe^{1/x}$. Here we could continue finding higher order terms, but it turns out that in this nice example, y is already a solution to the equation.

Ref: See Bender and Orszag

Qual : Jan 05, problem 1, Jan 04 problem 2