# The Method of Characteristics

This is a very large powerful topic, and in principle can be use to solve any quasilinear pde (where the the pde is linear in its highest derivative term). The method is the following. Presume you are interested in solving a pde like

$$u_t + F(u)_x = 0$$

Then one interpretation of this pde would be as a transport equation for u with speed $F'(u)$. In the simplest case this would be the linear transport equation

$$u_t + au_x = 0$$

where here $a$ is a constant. It is not hard to see here that the solution to the initial value problem $u(0, x) = f(x)$ is $u(x, t) = f(x - ct)$. Now lets see if I can construct this solution. Consider some curves in the $x, t$ plane $x(t)$ such that $\dot{x} = a$ then look at how u must change along these curves

$$\frac{dU(x(t), t)}{dt} = U_x x_t + U_t = aU_x + U_t = 0$$

Thus U is constant along these curves. Notice now that the curves here are $x(t) = at + x_0$, so $U(x, t) = U(at + x_0, t) = U(x_0, 0) = f(x_0)$. If we solve not for $x_0 = x - at$ we see that $U(x, t) = f(x - at)$. This example is trivial, but it illustrates the steps.

## 1 Burgers' Equation

In the first lecture we used conservation laws to derive Burgers' equation as a conservation law for density of traffic $u$.

$$u_t + uu_x = 0$$

Next we will discuss how to solve this equation with three types of initial data. $u(x, 0) = (x > 0)$, $u(x, 0) = (x < 0)$ and $u(x, 0) = (x < 1)(x > 0)$. Here the notation $(x < 0)$ returns 0 for the inequality false, and 1 for true, as in matlab. First, follow the steps of the previous problem, define curves by $\dot{x} = u(x(t), t)$. Then along these curves we see that the solution is constant. Thus we can solve for the curves, and get that $x(t) = u(x_0, 0)t + x_0$. At first glance it seems like we have solved our problem completely, however the characteristic curves $x(t)$ in this problem may or may not be parallel lines as in the linear transport equation. If we look at our three initial value problems we see in fact the lines are not parallel, and we need to deal with regions that they do not reach, and also where they intersect.

## 2 Shocks

If we want to solve this (Riemann) problem with $u(x, 0) = (x < 0)$ we see that the solution has two values specified by the characteristic curves in the region $t > 0, 0 < x < t$. Ie, the characteristics intersect in this region There are many physical interpretations of this phenomena, but we will just refer to it as a shock wave. Since we had discontinuous initial data, we can imagine that we could have a discontinuous solution. The question is, where to put the discontinuity. The answer comes from the conservation law. To derive burgers equation we had

$$\partial_t \int u = -\int F(u)_x$$

If we have a discontinuity then we can't have derivatives, so we write the integrated version

$$\int_t \int_{x_0}^{x_1} u = \int_t F(u)|_{x_0}^{x_1}$$

If we look then at an infinitesimal interval in $t$ and $x$ with $s = dx/dt$, we get a picture like that in Figure **??** This allows us to derive the conditions for a shock based on the conservation of u, to get what is called the

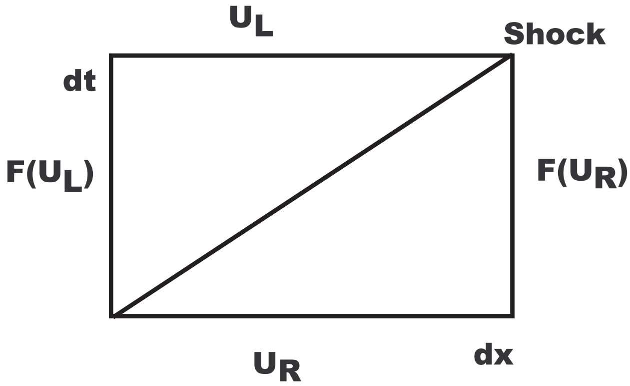


Figure 1: A cartoon illustrating how the Rankine Hugoniot Conditions are derived.

Rankine-Hugoniot conditions for the shock speed $s$

$$s = \frac{F(U_l) - F(U_r)}{(U_l - U_r)}$$

Now this is the only allowable shock speed, any other discontinuity will not conserve $u$. Thus we can solve this Riemann problem by inserting one shock in the region where characteristics intersect. Notice that the characteristics here all terminate in the shock.

# 3 Rarefaction

Look now at the problem of $u(x, 0) = (x > 0)$. Here the characteristics do not intersect, but rather there is a region from $0 < x < t$ where no characteristics reach. Now we could suppose that there is a shock here, but then what values should the solution take on either sides of the shock. The solution must be 0 on the left side, and one on the right side, or else we would have another discontinuity moving at an unallowable speed. Now we can find the characteristics in this region, and we see that they all emanate from the shock. Lax et al have derived a condition to rule out such solutions (See Leveque), which is called the Lax Entropy Condition. It says essentially that all characteristics must converge to a shock. We can motivate this condition by looking at a sequence of problems with initial data $u_\epsilon(x, 0) = (1 + tanh(x/\epsilon))/2$, and claiming that the riemann problem should be the limit of the solutions $u_\epsilon(x, t)$ as $\epsilon \to 0$. The Lax Entropy Condition eliminates the possiblility of a discontinuity in the region we are looking at. If there is no discontinuity, then the solution is continuous, and moreover is still constant along lines $x = at + c$. Since we know that these lines must pass through the origin, else we would have a discontinuity, we get that $c = 0$. Now we seek a continuous solution $u(x, t) = f(x/t)$, and we get the rarefaction wave $u = x/t$.

# 4 Bumps

The last set of initial data is basically the combination of the previous two. We can use the same method, and piece the previous two solutions together up to $t = 2$ where the rarefaction wave catches up with the shock. Here we must redetermine the shock speed by solving

$$s = \dot{x} = F(U_l) - F(U_r)/(U_l - U_r)$$

and substituting $U_l = x/t$ and $U_r = 0$ which gives the ode

$$dx/dt = 1/2x/t$$

which we solve to get the shock has equation $s(t) = at^{1/2}$ with $s(2) = 2$, so $a = \sqrt(2)$.

Ref: See Leveque, Finite Volume Methods for Hyperbolic Problems
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