

Perturbation Theory

For some problems, exact solutions are difficult to find, and it is easier to find a series of approximate solutions. The idea here is to find a small parameter in the problem, and then write a series expansion of the solution in that small parameter.

0.1 Example

Consider the following cubic

$$x^3 - 4.001x + 0.002 = 0$$

Finding the roots of this cubic is a messy problem, however we can notice here that this problem is very close to

$$x^3 - 4x = 0$$

which we can solve easily. This motivates us to write the solution as

$$x = \sum_{n=0}^{\infty} a_n \epsilon^n$$

and the equation as

$$x^3 - (4 + \epsilon)x + 2\epsilon = 0$$

where we can then plug in $\epsilon = 0.001$ to get the solution to the problem. Substituting this ansatz into the equation yields a system of equations for a_n

$$\begin{aligned} a_0^3 - 4a_0 &= 0 & \text{at } 0(1) \\ 3a_0^2 a_1 - 4a_1 - a_0 + 2 &= 0 & \text{at } 0(\epsilon) \\ & \dots \end{aligned}$$

Here we get immediately that $a_0 = 0, 2, -2$. Then we can solve the next equation for $a_1 = 1/2, 0, -1/2$. Thus we can write the roots $x = (0 + \epsilon/2, 2, -2 - \epsilon/2) + 0(\epsilon^2)$. Substituting $\epsilon = 0.001$ gives a good approximation for x . Notice it would not be difficult to do this to machine precision.

We can also use this same method to find solutions to ode's and pde's as series of functions. The technique is the same, and just like we had to only solve linear equations for a_i we only have to solve linear differential equations for these functions.

1 Linear Stability Analysis

After finding the first order solution by perturbation series expansion, one can ask is this solution stable. If so, then for small enough ϵ we may only care about the first order solution. Linear stability analysis asks the question, is the next order term bounded independent of ϵ , if so, then we say the solution is linearly stable, if not, then not linearly stable.

1.1 Example

Consider the following nonlinear PDE

$$\begin{aligned} u_t + u_x &= u^2 \\ u(x, 0) &= \epsilon f(x) \end{aligned}$$

If we look for an expansion $u = \sum \epsilon^n u_n(x, t)$ then we reduce this equation to a system of linear pde's. The first is

$$\begin{aligned} u_{1,t} + u_{1,x} &= 0 \\ u_1(x, 0) &= f(x) \end{aligned}$$

Where it is trivial to see the solution is $u_1(x, t) = f(x - t)$. Now to see if this solution is stable we look at the next order,

$$\begin{aligned}u_{2,t} + u_{2,x} &= f(x - t)^2 \\ u_2(x, 0) &= 0\end{aligned}$$

Here we can solve via the method of characteristics to get $u_2(x, t) = tf^2(x - t)$. Here we see that u_2 grows unbounded in time, so the solution u_1 is unstable. Note, we could have solved this problem at the start via the method of characteristics. The only reason here to do the perturbation expansion is to see that the linear solution is unstable.

Qual : Jan 04 problem 4
Famous Problem: Benjamin Feir instability
References: Keener