Exam 2 Material

Note: This is not an exhaustive list of problems, nor is it in the format of the exam. It is just some extra practice problems.

1. Where is the function $f(x,y) = \sqrt{x^2+y^2}$ continuous? Where is it differentiable? Find the linear approximation (tangent plane) to f at a point where it is differentiable.
   Solution: The function is not differentiable at the origin. An example tangent plane is at $(1,0)$, where the tangent plane is $z = x$

2. Find the directional derivative of the function $f(x,y) = \cos(x)\sin(y)$ at the point $(\pi/3, \pi/4)$ in the direction $(\sqrt{6}, \sqrt{2})$.
   Solution: $D_u f = \nabla f \cdot u = -\sqrt{2}/4$

3. Does the existence of a gradient at a point imply the existence of directional derivatives at that point? What about the reverse?
   Solution: If the gradient exists, the only directional derivatives that need exist are f_x and f_y, so no. In reverse, derivatives in other directions do not imply that the derivatives f_x and f_y exist.

4. For (x,y,z) on the surface $F(x,y,z) = 0$, with $F(x,y,z)$ differentiable, show that
   $$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1$$
   Solution: The chain rule applied to F gives, $\frac{\partial x}{\partial y} = -\frac{F_y}{F_x}$. Using the equivalent formula's for $\frac{\partial y}{\partial z}$ and $\frac{\partial z}{\partial x}$ gives the result.

5. Calculate $\frac{\partial}{\partial s}F(x(s), y(s))$ if $\frac{\partial F}{\partial x} = 5$, $\frac{\partial F}{\partial y} = 3$, $\frac{\partial x}{\partial s} = -1$ and $\frac{\partial y}{\partial s} = 2$.
   Solution: The chain rule gives $\frac{\partial}{\partial s}F = \frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s}$. Substituting in for everything gives $F_s = -5 + 6 = 1$

6. Find and classify all critical points of $F(x,y) = x^3y - x^2 - y^2$.
   Solution: $\nabla F = (3x^2y - 2x, x^3 - 2y) = 0$ gives three critical points $(0,0)$ and $(\pm(4/3)^{1/4}, \pm\frac{2}{3(4/3)^{1/4}})$. Here $D = f_{xx}f_{yy} - f_{xy}^2 = 4 - 12xy - 9x^4$, $D(0,0) = 4 > 0$, $f_y y = -2$ so $(0,0)$ is a maximum. $D(\pm(4/3)^{1/4}, \pm\frac{2}{3(4/3)^{1/4}}) = -16 < 0$ so the other two points are saddles.

7. Find the maximum of $F(x, y, z) = x^2 - y - z$ subject to the constraint $x^2 - y^2 + z = 0$.
Solution Use Lagrange Multipliers, defining $H(x, y, z, \lambda) = x^2 - y - z + \lambda(x^2 - y^2 + z)$ and setting $\nabla H = 0$. The maximum is at $(0, -1/2, 1/4)$.

8. Evaluate
$$\int_0^3 \int_2^4 x^2 y^2 dx dy$$
Solution 168 (a cool way to do this is to write it as the product of two integrals)

9. Find the new bounds if $\int_{-2}^{2} \int_0^{4-x^2} f(x, y) dy dx$ is written in the form $\int \int f(x, y) dx dy$.
Solution $\displaystyle\int_0^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} f(x, y) dx dy$

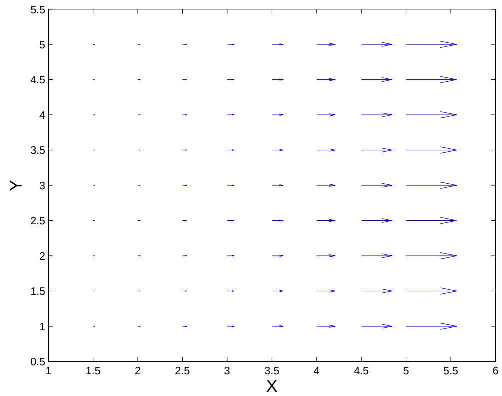
10. Find the average value of $F = \sqrt{x^2 + y^2}$ over the unit disc.
Solution It is easiest to do this in polar coordinates
$$Avg = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 r^2 dr d\theta = 2/3$$

11. Find the center of mass of the solid described by $0 < x < 1$ and $x^2 + y^2 + z^2 \leq 1$ with density function $\rho(x, y, z) = x^2 + y^2 + z^2$.
Solution We can get that $y_c = 0$ and $z_c = 0$ by observing the problem's symmetry. To get x_c use
$$x_c = \frac{\int \int \int x\rho dV}{\int \int \int \rho dV}$$
Use spherical coordinates to get the denominator is $2\pi/5$, the numerator is $\pi/6$, so $x_c = 5/12$. (Be careful here not to confuse the density function $\rho(x, y, z)$, with the radius in spherical coordinates, also denoted ρ)

12. Evaluate the volume of the region which is above the plane $z = 0$, outside of the cone $z = \sqrt{x^2 + y^2}$, and inside of the sphere $x^2 + y^2 + z^2 = 1$
   [i] Using an integral in cylindrical coordinates.
   [ii] Using an integral in spherical coordinates.
Solution The answer to both is $\frac{\sqrt{2}\pi}{3}$.

13. Consider the change of variables $u(x, y) = x^2 - y^2$, $v(x, y) = x^2 + y^2$, find the Jacobian of this transformation.
Solution Defining $\phi = (u, v)$, then $Jac(\phi) = u_x v_y - u_y v_x = 8xy$.

14. Draw a representation of the vector field which corresponds the potential $\phi = e^x$.
Solution



15. Calculate $\int_C \mathbf{F} \cdot d\mathbf{s}$ if C is the curve which connects $(0,1)$ to $(1,5)$ via $x(t) = t$ $y(t) = t^2 + 1$ with $t \in [0,1]$, and $\mathbf{F} =< x, \sqrt{y} >$.
Solution

$$\int_0^1 < t, \sqrt{t^2+1} > \cdot < 1, 2t > dt = \int_0^1 t + 2t\sqrt{t^2+1}dt = -\frac{1}{6} + \frac{4\sqrt{2}}{3}$$