Math 430: Formal Logic I
In-class worksheet for November 3rd, 5th, and 7th.

Some of these problems are on the problem set 9, due Monday, November 10th.

This worksheet is about the first-order structure $\mathcal{R}$ for the first-order language $L := \{P, M, O\}$ where P and M are binary function symbols, and O is a binary relation symbol. The universe of this structure is $\mathbb{R}$, $P^{\mathcal{R}}$ is the usual addition, $M^{\mathcal{R}}$ is the usual multiplication, and $O^{\mathcal{R}} := \{(a, b) \in \mathbb{R}^2 \mid a < b\}$ as the usual strict ordering.

1. For each of the following L-terms, write down the corresponding term function on $\mathcal{R}$ in a way that your 10th grade math teacher would like:
   $t_1(x, y) := P(P(x, y), y)$, $t_2(x) := M(x, x)$,
   $t_3(x, y, w) := P(P(M(x, M(y, y)), M(x, M(y, y))), M(x, z))$
2. For each of the following L-formulae, describe the corresponding relation on $\mathcal{R}$ in a way that your 10th grade math teacher would understand:
   $\phi_1(x) := \exists y\, M(y, y) \approx x,$
   $\phi_2(w) := ((M(w, w) \approx w) \wedge (\neg P(w, w) \approx w))$
   $\phi_3(x, y) := \exists w\, (\phi_2(w) \wedge O(P(P(x, w), w), Y))$
   $\phi_4(x, z) := \exists w\, (\phi_2(w) \wedge ((O(w, x) \wedge z \approx P(x, w)) \vee (O(x, w) \wedge z \approx P(x, x))))$
3. Which of the following L-sentences are satisfied by $\mathcal{R}$?
   $\forall x\, \forall y\, \forall z\, P(P(x, y), z) \approx P(x, P(y, z))$
   $\exists x\, \exists y\, \neg M(x, y) \approx M(y, x)$
   $\exists x\, \exists y\, (O(x, y) \wedge \exists z\, P(y, M(z, z)) \approx x)$
4. Describe all term functions on $\mathcal{R}$ in a way that your 10th grade math teacher would like.
5. Give several examples of definable functions that are not term functions.
6. List as many definable elements of $\mathbb{R}$ as you can.
7. Show that is a finite subset $S \subset \mathbb{R}$ is definable in $\mathcal{R}$, then for every element $a \in S$, $\{a\}$ is definable in $\mathcal{R}$. Now add some more elements to your list of definable elements.
8. Write down some L-sentences that express some familiar properties of the reals. What properties are you having trouble expressing?
9. Given an L-term $t(x)$, can you write down a formula asserting that $t^{\mathcal{R}}$ is continuous at zero? that $t^{\mathcal{R}}$ is differentiable at zero?
10. What are the automorphisms of $\mathcal{R}$?
11. What are the endomorphisms of $\mathcal{R}$?
12. Is any symbol of L definable from the others?