

**Ma502: Metamathematics I**  
**Final Exam**

Chapters 2-6, 10, and 12 of Ebbinghaus, Flume, and Thomas (EFT) and  
Chapters 0, 1, and 3 of Kaye.

Hand your solutions to me or slide them under my office door SEO 716 **no  
later than noon on Wednesday, December 12th.**

While taking this exam, you may use our textbooks, EFT and Kaye, and anything you have hand-written during this course: lecture notes, homework solutions, etc. You may not use other books or internet, and you may not collaborate. You should spend no more than 6 hours on this exam; feel free to take a couple 10-15 minute breaks. Anything written after 6 hours should be clearly marked as such, and will receive substantially less credit. The exam questions will be on the second page so that you can print them without looking. Should you have any questions, procedural or mathematical, feel free to ask me. Good luck!

The problems are roughly in the order of increasing difficulty.

1. (10 points) Let  $S$  be a symbol set,  $M$  an  $S$ -structure,  $\beta$  an  $M$ -assignment,  $a$  an element of the universe of  $M$ ,  $\phi(x)$  an  $S$ -formula with one free variable  $x$ ,  $R$  a binary relation symbol in  $S$ ,  $f$  a unary function symbol in  $S$ .

(a)  $\beta(x)$

- i. might be true or false.
- ii. is an  $S$ -term.
- iii. is an element of  $M$ .
- iv. is meaningless.

(b)  $f(a)$  is

- i. as  $S$ -term.
- ii. an element of  $M$ .
- iii. might be satisfied or not satisfied by an  $S$ -structure.
- iv. meaningless.

(c)  $f^M(\beta(x))$  is

- i. as  $S$ -term.
- ii. might be satisfied or not satisfied by an  $S$ -structure.
- iii. an element of  $M$ .
- iv. meaningless.

(d)  $M \models \phi(x)$

- i. might be true or false.
- ii. is not well-defined as written, but is a standard abbreviation for the well-defined statement  
"there is some  $M$ -assignment  $\alpha$  such that  $(M, \alpha) \models \phi(x)$ ."
- iii. is meaningless.

(e)  $M \models \phi(a)$

- i. might be true or false.
- ii. is not well-defined as written, but is a standard abbreviation for the well-defined statement  
"there is some  $M$ -assignment  $\alpha$  such that  $\alpha(x) = a$  and  $(M, \alpha) \models \phi(x)$ ."
- iii. is meaningless.

(f)  $(M, \beta) \models \phi$

- i. might be true or false.
- ii. is not well-defined as written, but is a standard abbreviation for the well-defined statement  
"there is some element  $b$  of  $M$  such that  $M \models \phi(b)$ ."
- iii. is meaningless.

- (g)  $R(x, x)$
- i. might be true or false.
  - ii. might be satisfied or not satisfied by an  $S$ -structure.
  - iii. is an  $S$ -term.
  - iv. is meaningless
- (h)  $R^M(a, a)$
- i. might be true or false.
  - ii. might be satisfied or not satisfied by an  $S$ -structure.
  - iii. is an element of  $M$ .
  - iv. is meaningless
- (i)  $R^M(x, x)$
- i. might be true or false.
  - ii. might be satisfied or not satisfied by an  $S$ -structure.
  - iii. is meaningless
  - iv. is an  $S$ -formula

2. (10 points) Give a first-order axiomatization for the following classes, or show that it is impossible:

- (a) the class of *torsion groups*, i.e. groups where every element has finite order;
- (b) the class of *torsion-free groups*, i.e. groups where no element has finite order.

(The *order* of an element  $g$  of a group is the least positive integer  $k$  such that  $g^k = 1$ ; if no such  $k$  exists, the element is said to have infinite order.)

3. (10 points) Here are three functions:

- (a)  $f(x) = 2^x$ .
- (b)  $g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  is an injection.
- (c)  $h(x, y) = 1$  if the greatest common divisor of  $x$  and  $y$  is 1, otherwise  $h(x, y) = 0$ .

Write a register machine program that computes one of them, show that another is partial recursive, and give a  $\Delta_0$  formula representing the third. You are free to choose what to do to which function, and unwise choice will make this problem much harder. Please, include **details!**

4. (15 points) Let  $S = \{R\}$  be a symbol set containing one binary relation. Let  $T =$

$$\begin{aligned} & \{\forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz), \\ & \forall x \neg Rxx, \forall x \forall y (Rxy \vee Ryx \vee x = y), \\ & \forall x \exists y (Rxy \wedge \forall z Rxz \rightarrow (Ryz \vee y = z)), \\ & \forall x \exists y (Ryx \wedge \forall z Rzx \rightarrow (Rzy \vee y = z))\} \end{aligned}$$

For example,  $(\mathbb{Z}, <) \models T$ . Show that  $T_{\vdash}$  is decidable. Hint: use the back-and-forth characterization of elementary equivalence to show that  $T_{\vdash}$  is complete.

5. (15 points) Modify the proof of the completeness theorem to prove a baby version of the Omitting Types Theorem: Fix a symbol set  $S$ , suppose  $T$  is a set of  $S$ -sentences,  $p(x)$  is a set of  $S$ -formulae with one free variable  $x$ , and  $T \cup p(x)$  is consistent. Suppose also that  $p$  is *not isolated modulo*  $T$ , i.e. no non-trivial  $S$ -formula implies all formulae in  $p$  modulo  $T$ : i.e. there is no  $\phi(x)$  such that  $T \cup \phi(x)$  is consistent and for all  $\psi(x) \in p(x)$

$$T \vdash (\phi(x)) \rightarrow \psi(x)$$

Show that there is an  $S$ -structure  $M \models T$  such that no element of  $M$  realizes  $p$ , i.e. for all  $a \in M$  there is a formula  $\psi(x) \in p(x)$  such that  $M \models \neg \psi(a)$ .

Don't try to do this with compactness - it won't work.