Ma502: Metamathematics I

9th homework set, due monday, november 12th.

Chapter 10 of Ebbinghaus, Flume, and Thomas (EFT) and Chapter 0 of Kaye.

Bring your solutions to class, or slide them under the door of SEO716.

1. Show that there exists a function that isn't partial-recursive.
2. Prove Proposition 0.8 on p. 7 of Kaye.
3. Prove Proposition 0.9 on p. 7 of Kaye.
4. The *Ackermann function* $A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined by

   $$A(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ A(m-1, 1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m-1, A(m, n-1)) & \text{if } m > 0 \text{ and } n > 0. \end{cases}$$

   You should convince yourself that this function is well-defined for all inputs, and that it is partial recursive.

   Show by induction on the definition of primitive recursive functions that for every primitive recursive function $f : \mathbb{N}^k \to \mathbb{N}$ there is a natural number C such that for all $a_1, \ldots a_n$ we have $f(a_1, \ldots a_n) \leq A(C, max(a_1, \ldots a_n))$. Thus, the Ackermann function is an example of an everywhere-defined partial recursive function that isn't primitive recursive.

   Copious hints:

   (a) This should be obvious, but the first thing you should do is try to compute a few values: what is $A(1,1)$? $A(3,3)$?

   (b) Find mathematical formulae (polynomials, exponentials...) for $A(1,n)$, $A(2,n)$, and $A(3,n)$.

   (c) Show that $A(c,n) > n$ for all c, n.

   (d) Show that for a fixed m, $A(m,n)$ is increasing in n. I.e. For all m, n, n', if $n \leq n'$ then $A(m,n) \leq A(m,n')$.

   (e) The hard induction step is primitive recursion: suppose that f is obtained by primitive recursion from g and h; show that if $c > 2$ works for both g and h, then $c+2$ will work for f.

5. Ask an interesting question and try to answer it.