Definitions for Unstable Formula Theorem:

2.28 Definition. The formula $\phi(\bar{x};\bar{y})$ has the *order property* if in some model of T there exist infinite sequences $\{\bar{a}_i : i < \omega\}$ and $\{\bar{b}_i : i < \omega\}$ such that $\models \phi(\bar{a}_i;\bar{b}_j)$ iff $i < j$.

We want to generalize that just a bit to "L-order property" for an arbitrary linear ordering L. By compactness, the statement doesn't depend on the infinite ordering chosen.

The following refines Baldwin's Definition III.1.15 on p. 58, which appears as (4) below. A phi-m-type is just an phi-type.

DEFINITION 2.1: (1) The φ-m-type p is $\psi(\bar{y};\bar{c})$-*defined* if $\varphi(\bar{x};\bar{a}) \in p \Rightarrow \models\psi[\bar{a};\bar{c}]$ and $\neg\varphi(\bar{x};\bar{a}) \in p \Rightarrow\ \models\neg\psi[\bar{a};\bar{c}]$.

(2) The φ-m-type p is $(\psi(\bar{y};\bar{z}), A)$-*definable* if there is a $\bar{c} \in A$ such that p is $\psi(\bar{y};\bar{c})$-defined.

(3) The φ-m-type p is A-*definable* if there is a ψ for which p is (ψ, A)-definable.

(4) The m-type p is A-definable if $p \restriction \varphi$ is A-definable for every φ.

And finally, Shelah's R-ranks, an amalgam of definitions VII.2.2 and III.1.10 in Baldwin:

DEFINITION 1.1: Let p be an m-type, Δ a set of m-formulas, λ a cardinal (possibly finite), or $\lambda = \infty$. We define the *rank* $R^m(p, \Delta, \lambda)$ or $R^m[p, \Delta, \lambda]$ by defining inductively when $R^m(p, \Delta, \lambda) \geq \alpha$, α an ordinal.

(1) $R^m(p, \Delta, \lambda) \geq 0$ when p is a (consistent) type. (When p is inconsistent we stipulate $R^m(p, \Delta, \lambda) = -1$.)

(2) $R^m(p, \Delta, \lambda) \geq \delta$ for δ a limit ordinal if $R^m(p, \Delta, \lambda) \geq \alpha$ for all $\alpha < \delta$.

(3) $R^m(p, \Delta, \lambda) \geq \alpha + 1$ if for all $\mu < \lambda$ and all finite $q \subseteq p$ there are types $\{q_i\}_{i \leq \mu}$ which are Δ-m-types (i.e., m-types whose formulas are all of the form $\varphi(\bar{x};\bar{a})$ or $\neg\varphi(\bar{x};\bar{a})$ where $\varphi(\bar{x};\bar{y})$ is in Δ) such that:

(i) for $i \neq j$ there is a formula φ such that $\varphi \in q_i$, $\neg\varphi \in q_j$ (or vice versa). In this case we say that q_i and q_j are explicitly contradictory.

(ii) $R^m(q \cup q_i, \Delta, \lambda) \geq \alpha$ for all $i \leq \mu$.

(4) Now if $R^m(p, \Delta, \lambda) \geq \alpha$ but not $R^m(p, \Delta, \lambda) \geq \alpha + 1$ we say $R^m(p, \Delta, \lambda) = \alpha$. (It is easy to see by induction on α that $R^m(p, \Delta, \lambda) \geq \alpha$ implies $R^m(p, \Delta, \lambda) \geq \beta$ for all $\beta \leq \alpha$.) If $R^m(p, \Delta, \lambda) \geq \alpha$ for all α we define $R^m(p, \Delta, \lambda) = \infty$.

This is a "phi-kappa-tree above p" for a formula phi, a type p, and a cardinal kappa.
If unspecified, p is { x=x }.

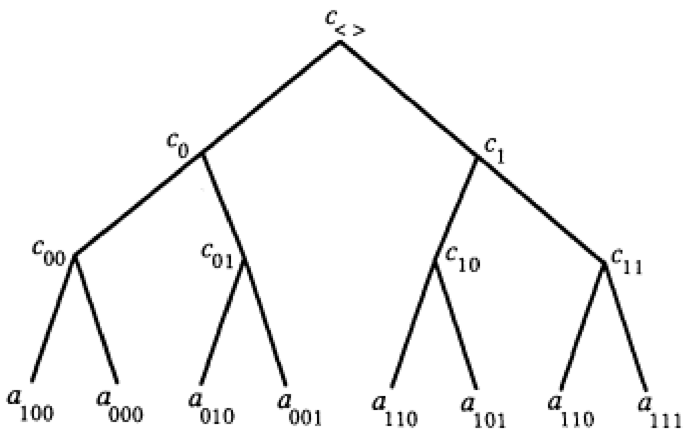

Fig. 1. $\Gamma_{\langle\,\rangle}(\phi, 2)$

to L additional constant symbols $\overline{a}_\tau, \tau \in 2^\kappa$ and $\overline{c}_s, s \in 2^{<\kappa}$.

$$\Gamma_p(\phi, \kappa) = \{p(\overline{a}_\tau) : \tau \in 2^\kappa\} \cup \{\phi(\overline{a}_\tau; \overline{c}_{\tau|i})^{\tau(i)} : i < \kappa\}.$$

Here $p(\overline{a}_\tau)$ denotes $\{\psi(\overline{a}_\tau) : \psi \in p\}$ and $\phi^{\tau(i)}$ denotes ϕ or $\neg\phi$ depending on whether $\tau(i)$ is 0 or 1.

The collection $\Gamma_p(\phi, \kappa)$ is consistent just if there is a complete binary tree of height κ of extensions of p by instances of ϕ such that each path is consistent but the paths are pairwise contradictory. Note we can easily extend our definition to $\Gamma_{\langle\,\rangle}(\phi, \kappa)$ by letting $\langle\,\rangle$ denote the empty type. The following diagram illustrates the definition of $\Gamma_p(\phi, \kappa)$. The $\overline{a}_\tau$ in the definition correspond to paths through the tree; the $\overline{c}_s$ correspond to nodes.

The 10 equivalent conditions:

1 big phi-stone space for all lambda
2 big phi stone space for one infinite lambda
3w order property
3f L-order property for all finite linear orders L
3w* L-order property where L is omega with the opposite ordering
4 phi-trees of every finite height
5 phi-trees of every ordinal height
6 R^m(x=x, \phi, \infty) = \infty
7 R^m(x=x, \phi, 2) \geq \omega
8 there is a type which is not definable
9 for every psi, there is a phi-type which is not psi-definable
10 no phi-stone space has infinite Cantor-Bendixon rank