MATH 320: HOMEWORK 3

Due on Wednesday, September 18

In the assigned Hefferon textbook, do exercises 1.42 and 1.43 on page 86.

1) Let V be a vector space. Prove the following:

(1) The zero element $0 \in V$ is unique.
(2) For any $v \in V$, the additive inverse $-v \in V$ is unique.

2) Using the definition, prove that the space of solutions S to the linear system:

$$\begin{aligned} 3x + y - z &= 0 \\ y + z &= 0 \end{aligned}$$

is a vector space. Write down a basis for the vector space S. (Hint: we proved a theorem while studying homogeneous linear systems which tells us a basis, you may use this theorem if you can cite it properly).

3) Find a basis for $\mathbb{R}^3$ such that all coordinates entries of each basis element are ± 1.

4) Let L be the map $L : \mathbb{R}^3 \to \mathbb{R}^2$ defined by,

$$L((x, y, z)) = (x + y - z, x - z)$$

(1) Show that L is linear.
(2) With respect to the standard basis of $\mathbb{R}^3$ and $\mathbb{R}^2$, find a matrix A such that

$$L((x, y, z)) = A\vec{v},$$

where $\vec{v}$ is the column vector

$$\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(3) Use Gauss-Jordan elimination to put the matrix A in reduced row echelon form.
(4) Find the set of all vectors $v \in \mathbb{R}^3$ such that $L(v) = 0$. (Hint: the row reduced echelon form can help you do this).

Date:

5) Prove or disprove: the map $L : \mathbb{R} \to \mathbb{R}$ defined by

$$L(x) = x^2$$

is linear.

6) Consider the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

as a linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$. Here, A is the matrix of L with respect to the standard basis.

(1) Where does A send the column vector

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2?$$

(2) Draw a picture of the image of the unit square with vertices $(0, 0), (1, 0), (1, 1)$ and $(0, 1)$ under the mapping A.

(3) What is the area of the image of the unit square under the map A?

(4) Let S be a square in $\mathbb{R}^2$ with area equal to α. Using your answer to (3), prove that the area of the image of S under the matrix A is also equal to α.

For interested parties: If you are feeling saucy and the previous problems didn't satisfy your curiosity, use the change of variables formula from multi-variable calculus to prove that for any open set $S \in \mathbb{R}^2$ with smooth boundary such that the area of S is equal to α, the set $A(S)$ also has area equal to α.