MATH 320: HOMEWORK 4

Due on Friday, October 11

Question Zero: Give the precise mathematical definition of a basis of a vector space. If this is not done perfectly, this homework will be assigned a grade of zero and no other problems will be considered.

1) Find a general formula for the inverse of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

satisfying $ad - bc \neq 0$.

2) Consider the followings bases of $\mathbb{R}^2$:

$$\mathfrak{B} := \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

and

$$\mathfrak{B}' := \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

(1) Prove that $\mathfrak{B}$ and $\mathfrak{B}'$ are bases.
(2) Compute the change of basis matrix Ψ from $\mathfrak{B}$ to $\mathfrak{B}'$.
(3) Write the linear transformation

$$\begin{aligned} L : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x, y) &\longmapsto (2x + y, 3x - 2y) \end{aligned}$$

with respect to the bases $\mathfrak{B}$ and $\mathfrak{B}'$.
(4) Verify the change of basis formula, namely that $[L]_{\mathfrak{B}}$ and $[L]_{\mathfrak{B}'}$ are related by conjugation by Ψ.

3) Let $\mathbb{R}_2[t]$ be the vector space of polynomials of degree two with real coefficients.

(1) Compute the span of the subset $S = \{1 + t, 1 + t^2, 2 + t + t^2\}$. Show that the elements of S are not linearly independent.
(2) Find a subspace $M \subset \mathbb{R}_2[t]$ such that $\mathbb{R}_2[t] \simeq \mathrm{Span}(S) \oplus M$.

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(3) Compute the kernel and image of the linear map,

$$L : \mathbb{R}_2[t] \to \mathbb{R}_2[t]$$
$$(a_0 + a_1 t + a_2 t^2) \longmapsto a_0 + (a_0 + a_1 + a_2)t^2.$$

(4) Verify the rank-nullity theorem, namely that $\text{Dim}\,(\mathbb{R}_2[t]) = \text{Dim}(\text{Ker}(L)) + \text{Dim}(\text{Im}(L))$.

4) Let V be a vector space. Prove that elements $v_1, ..., v_n \in V$ are linearly independent if and only if

$$\text{Span}(v_1, ..., v_n) = \text{Span}(v_1) \oplus ... \oplus \text{Span}(v_n).$$

5) Assume that $L : V \to V$ is a linear map and $v_1, ..., v_k \in V$ are linearly dependent. Prove that $L(v_1), ..., L(v_k)$ are linearly dependent.

6) Find the kernel and image of the linear map,

$$L : \mathbb{R}^4 \to \mathbb{R}^4$$
$$(x, y, z, w) \longmapsto (x + y + z + w, y - z + w, x, x + 2y + 2w).$$