MATH 320: HOMEWORK 6

Due on Friday, November 22

1) Let $A \in M_n(\mathbb{R})$ be an $n \times n$ matrix satisfying $A^t = A$ and such that there exists an invertible matrix $S \in M_n(\mathbb{R})$ satisfying $A = S^t S$. A matrix satisfying these properties is *symmetric* and *positive definite*. Given column vectors $v, w \in \mathbb{R}^n$, define a pairing,

$$\langle v, w \rangle_A := v^t A w,$$

where the right hand side is defined using matrix multiplication.

(1) Prove that $\langle v, w \rangle_A$ defines an inner product on $\mathbb{R}^n$.
(2) Suppose that A is given by

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

Find a basis of $\mathbb{R}^2$ which is orthonormal with respect to $\langle\ ,\ \rangle_A$.

2) Let $A \in M_n(\mathbb{C})$ be an $n \times n$ matrix satisfying $A^* = A$ and such that there exists an invertible matrix $S \in M_n(\mathbb{C})$ satisfying $A = S^* S$. A matrix satisfying these properties is *Hermitian* and *positive definite*. Given column vectors $v, w \in \mathbb{C}^n$, define a pairing,

$$\langle v, w \rangle_A := v^t A \overline{w},$$

where the right hand side is defined using matrix multiplication.

(1) Prove that $\langle v, w \rangle_A$ defines an inner product on $\mathbb{C}^n$.
(2) Suppose that A is given by

$$A = \begin{pmatrix} 1 & 2i \\ -2i & 5 \end{pmatrix}.$$

Find a basis of $\mathbb{C}^2$ which is orthonormal with respect to $\langle\ ,\ \rangle_A$.

3) Consider the following basis $\mathfrak{B}$ of $\mathbb{R}^3$,

$$\mathfrak{B} := \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\}.$$

(1) Using Gram-Schmidt orthogonalization, use the above basis to construct an orthonormal basis of $\mathbb{R}^3$ with respect to the usual dot product.

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(2) Find an inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^3$ such that the basis $\mathfrak{B}$ is orthonormal with respect to $\langle\ ,\ \rangle$.

4) Consider the matrix,

$$O = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

(1) Compute $\mathrm{Det}(O)$.
(2) Show that $OO^t = \mathbb{I}$ where $\mathbb{I}$ is the 3×3 identity matrix.
(3) Show that for all column vectors $v, w \in \mathbb{R}^3$,

$$O(v) \cdot O(w) = v \cdot w,$$

where $v \cdot w$ is the usual Euclidean dot product of the vectors v and w.
(4) If e_1, e_2 and e_3 is any orthogonal basis of $\mathbb{R}^3$ with respect to the Euclidean dot product, show that $O(e_1), O(e_2)$ and $O(e_3)$ is also an orthogonal basis.

5) Consider the determinant on 2×2 matrices as a map,

$$\begin{aligned} \mathrm{Det} : \mathbb{R}^2 \times \mathbb{R}^2 &\to \mathbb{R} \\ (v, w) &\mapsto \mathrm{Det}(v\ w), \end{aligned}$$

where $v, w \in \mathbb{R}^2$ are column vectors and $(v\ w)$ is the 2×2 matrix whose columns are given by the vectors v and w.

(1) Which of the properties of an inner product does Det satisfy?
(2) Find a 2×2 matrix D such that

$$\mathrm{Det}(v, w) = v^t D w.$$

(3) Consider the matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Show that,

$$\mathrm{Det}(v, J(w))$$

defines an inner product on $\mathbb{R}^2$. Do you recognize which inner product this is?

Remarks: the mapping Det is a special example of a *symplectic form* on a vector space. The matrix J is a special case of a *complex structure* on a vector space. The vector space $\mathbb{R}^2$ equipped with J and Det, hence also with an inner product by (3), is the simplest example of something known as a Hermitian vector space. These are an incredibly important and very special type of vector space, and form the basis of something known as the study of *Kähler* geometry. Much of my personal research concerns the study of these types of objects; in fact, one of the Clay Millennium problems in mathematics (whose solution is worth 1 million USD) is concerned with the structure of certain geometrical objects which arise in Kähler geometry.