HARMONIC MAPS - FROM REPRESENTATIONS TO HIGGS BUNDLES

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Abstract. The theory of Higgs bundles, initiated by Hitchin, has been instrumental in understanding the topology and geometry of character varieties. In addition, this gauge theoretic viewpoint provides a wealth of revealing, and puzzling, extra structure on the character variety. The key to the correspondence between representations of surface groups and Higgs bundles is the existence of an equivariant harmonic map from the universal cover of the surface to the symmetric space associated to a reductive Lie group G. The purpose of this note is to introduce the theory of Higgs bundles, with a strong emphasis put on the role of harmonic maps. We will begin with an overview of Fricke-Teichmüller theory via harmonic maps, and then explain how Higgs bundles generalize this point of view from $\mathrm{PSL}(2,\mathbb{R})$ to other Lie groups. We will focus exclusively on the linear groups $\mathrm{SL}(n,\mathbb{C})$, though it should be noted that the theory proceeds in an analogous fashion for any reductive Lie group.

Throughout, Σ is a fixed smooth, oriented, closed surface of genus greater than one. A base point $x \in \Sigma$ is fixed and let $\pi := \pi_1(\Sigma, x)$.

1. Fricke-Teichmüller theory via harmonic maps

Select hyperbolic metrics h_0 and h on Σ. In local isothermal coordinates $z = x^1 + ix^2$ and $w = y^1 + iy^2$ for h_0 and h respectively,

$$h_0 = e^{2u}|dz|^2$$

and

$$h = e^{2v}|dw|^2.$$

For any C^1-mapping $f : \Sigma \to \Sigma$, define the energy (or action functional):

$$\mathcal{E}(f) = \frac{1}{2}\int_\Sigma \|df\|^2 dV_{h_0}.$$

Above, we view the differential of f as a section $df \in \Gamma(T^*\Sigma \otimes f^*T\Sigma)$ and the norm $\|df\|^2$ (the *energy density*) is computed using the metrics h_0 and h, explicitly in local coordinates:

$$\|df\|^2 = h_0^{ij}\partial_i f^\alpha \partial_j f^\beta h_{\alpha\beta}(f).$$

Here, we employ Einstein summation convention with matching upper and lower indices being summed. Latin indices refer to domain variables and Greek indices to range variables; lastly,

$$f^\alpha = f \circ y^\alpha.$$

Definition 1.1. *A C^1-mapping $f : \Sigma \to \Sigma$ is harmonic if it is a critical point for the energy. Namely, given any C^1-variation through C^1-mappings $f_t : \Sigma \to \Sigma$, the first variation vanishes:*

$$\frac{d}{dt}\mathcal{E}(f_t)|_{t=0} = 0.$$

Throughout the rest of the notes, we will ignore all issues of regularity and assume all maps to be C^∞. This is not a major restriction for our purposes, the regularity theory for semi-linear elliptic PDE will guarantee the harmonic maps we meet are even real-analytic.

A smooth map $f : \Sigma \to \Sigma$ which is harmonic necessarily satisfies the Euler-Lagrange equations

$$\Delta_{h_0} f^\gamma + \Gamma^\gamma_{\alpha\beta}(f)\partial_i f^\alpha \partial_j f^\beta h_0^{ij} = 0 \tag{1.1}$$

for $\gamma = 1, 2$.

In these equations, Δ_{h_0} is the Laplace-Beltrami operator for the metric h_0 and the $\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols for the target metric h. This is a semi-linear system of elliptic partial differential equations; if this sounds a bit arcane, for our purposes it assures that solutions to this equation have very strong regularity and uniqueness properties.

Observe that everything said above makes sense replacing Σ by a pair of closed Riemannian manifolds (M, g) and (N, h). Harmonic maps in this setting were formally introduced by Eells and Sampson [ES64] culminating in:

Theorem 1.2. *Let (M, g) and (N, h) be closed, smooth Riemannian manifolds such that (N, h) has negative sectional curvature. Then in every homotopy class of maps $[M \to N]$ there exists a smooth harmonic map $f : (M, g) \to (N, h)$. Furthermore, provided f does not map onto a closed geodesic in N, it is unique.*

Remark: The existence statement above is due to Eells-Sampson, while the uniqueness is due to Hartman [Har67].

In our setting, The above Theorem becomes:

Theorem 1.3. *There exists a unique harmonic map $f : (\Sigma, h_0) \to (\Sigma, h)$ homotopic (even isotopic) to the identity.*

In conformal coordinates, the Euler-Lagrange equations (1.1) take the form

$$\frac{\partial f}{\partial z \partial \overline{z}} + 2e^{-v}\frac{\partial}{\partial z}\left(e^{v(f)}\right)\frac{\partial f}{\partial z}\frac{\partial f}{\partial \overline{z}} = 0.$$

The form of this equation exposes a crucial invariance:

- In the case that the domain is a surface, the harmonic map depends only on the conformal class of the metric. This follows since the metric h_0 appears nowhere, only the conformal coordinate z.

For this reason, a harmonic map from a surface is linked to the holomorphic geometry of the surface. This is made precise by the following Proposition due to Hopf [Hop54],

Proposition 1.4. *Define a quadratic differential by,*

$$\phi := \phi(z)dz^2 = h\left(\frac{\partial f}{\partial z}, \frac{\partial f}{\partial z}\right)dz^2.$$

If f is harmonic, then ϕ is holomorphic.

Written in the conformal coordinates,

$$\phi(z) = e^{2v(f(z))}\frac{\partial f}{\partial z}\overline{\frac{\partial f}{\partial \bar{z}}}.$$

The ϕ associated to a harmonic map f is called the Hopf differential. This defines a map, which depends on a choice of conformal structure $\sigma \in \mathcal{T}$ in the Teichmüller space of isotopy classes of conformal structures on the surface:

$$\begin{aligned} N_\sigma : \mathcal{F} &\longrightarrow QD(\Sigma, \sigma) := H^0((\Sigma, \sigma), K^2) \\ h &\longmapsto \text{Hopf differential of unique harmonic map.} \end{aligned}$$

Here we make the distinction between the Tiechmüller space $\mathcal{T}$ and the Fricke space $\mathcal{F}$ of isotopy classes of hyperbolic metrics on Σ. The two are only identified after making the transcendental identification afforded by the Köebe-Poincaré uniformization Theorem.

The fundamental theorem which allows one to "do" Teichmüller theory with this approach is due to the efforts of many mathematicians, we mention Wolf [Wol89] and refer to the references therein:

Theorem 1.5. *For each $\sigma \in \mathcal{T}$, the map*

$$N_\sigma : \mathcal{F} \longrightarrow H^0((\Sigma, \sigma), K^2)$$

is continuous, injective and proper. Hence it is a homeomorphism since the latter space of sections is a vector space.

Remark: In fact, the mapping is real analytic (due to the fact that the harmonic map depends real analytically on parameters) for the real analytic structure on the Fricke space induced by its incarnation as a component of the character variety

$$\chi(\pi, \mathrm{PSL}\,(2, \mathbb{R})) = \mathrm{Hom}\,(\pi, \mathrm{PSL}(2, \mathbb{R}))\,/\mathrm{PSL}(2, \mathbb{R}).$$

The proof outlines as follows: the continuity follows from the well-posedness of solutions to the harmonic map equations, i.e. solutions depend continuously on the data. The injectivity follows from an easy application of the maximum principle, once an equation has been concocted to which it can be applied. The properness is the only piece requiring specific consideration and can be found in the paper of Wolf [Wol89]. Morally, it is proper because it takes more "energy" to stretch a hyperbolic surface onto one which far away from it in the Fricke space.

Remarkably, Hitchin [Hit87], Simpson [Sim92], Donaldson [Don87], and Corlette [Cor88] discovered that this is part (a base case!) of a profound correspondence:

{ X closed, Kähler manifold }$\longrightarrow$ { Reductive representations of $\pi_1(X)$ into a reductive Lie group } $\longrightarrow$ { Holomorphic objects over X }.

The particularly ingeniuous leap stems from the fact that in our above discussion of harmonic maps and Fricke-Teichmüller theory, we are missing part of the holomorphic objects over X alluded to in the diagram above. These holomorphic objects are named Higgs bundles. The Hopf differential is a remnant of the Higgs bundle which we will construct for any reductive representation $\rho : \pi \to \mathrm{SL}(n, \mathbb{C})$.

Before we tackle this, we must recall the basic calculus of vector bundles.

2. Calculus on Vector Bundles: metrics and connections on bundles

A thorough reference for the following material is Kobayashi's book [Kob87].

Let M be a smooth, closed manifold and $V \to M$ a smooth real or complex vector bundle over M. We denote the $C^\infty(M)$-module of smooth sections of V by $\Gamma(V)$.

Definition 2.1. *A connection (or covariant derivative) is a first order differential operator*

$$\nabla : \Gamma(V) \to \Gamma(T^*M \otimes V)$$

satisfying:

$$\nabla(s + s') = \nabla s + \nabla s'$$
$$\nabla f s = df \otimes \nabla s + f \nabla s$$

for all $s, s' \in \Gamma(V)$ and $f \in C^\infty(M)$.

Given two connections ∇, ∇', the difference satisfies

$$\nabla - \nabla'(fs) = f(\nabla - \nabla')s.$$

Hence, $\nabla - \nabla' \in \Omega^1(\mathrm{End}(V))$ where $\Omega^1(\mathrm{End}(V))$ is the space of one-forms on M with values in the endomorphisms of V. In slightly fancy language, the space of connections on V, denoted $\mathcal{A}(V)$, is an affine space with underlying vector space of translations $\Omega^1(\mathrm{End}(V))$.

Given $\nabla \in \mathcal{A}(V)$, there exists a skew-symmetric extension called the exterior covariant differential operator associated to ∇ :

$$d^\nabla : \Omega^k(V) \to \Omega^{k+1}(V)$$

where $\Omega^k(V)$ is the space of exterior differential k-forms with values in V. It is defined by enforcing the graded Leibniz rule: given $\omega \in \Omega^k(M)$ and $s \in \Gamma(V)$,

$$d^\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge \nabla s.$$

The introduction of this operator allows for a concise definition of the *curvature* of a connection,

Definition 2.2. *Given a connection $\nabla \in \mathcal{A}(V)$, the curvature 2-form is defined by*

$$F^\nabla := d^\nabla \circ \nabla \in \Omega^2(End(V)).$$

Over a trivializing open set $U \subset M$, a section may be written $s = s^i e_i$. The action of a connection ∇ is

$$\nabla s = (ds^j + s^i A_i^j) e_j,$$

where the connection coefficients A_i^j (relative to U) comprise a matrix of 1-forms on U. Then the curvature of ∇ acts via

$$F^\nabla(s) = s^i(dA_i^k + A_j^k \wedge A_i^j) e_k.$$

Now suppose M is a complex manifold and V is a smooth, complex vector bundle. The complexified co-tangent bundle of M splits into types

$$T^*M \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}M \oplus T^{(0,1)}M.$$

Definition 2.3. *A pseudo-connection (or Cauchy-Riemann, or del-bar operator) is a first order differential operator*

$$\overline{\partial}_V : \Gamma(V) \to \Gamma(T^{(0,1)}M \otimes V)$$

such that

$$\overline{\partial}_V(s+s') = \overline{\partial}_V s + \overline{\partial}_V s'$$
$$\overline{\partial}_V(fs) = \overline{\partial} f \otimes s + f\overline{\partial}_V s.$$

This operator also has a graded skew-commutative extension to an operator which by abuse of notation we write the same way,

$$\overline{\partial}_V : \Omega^{p,q}(V) \to \Omega^{p,q+1}(V).$$

Similarly to connections, pseudo-connections are an affine space with underlying vector space of translations $\Omega^{0,1}(\mathrm{End}(V))$. The following integrability condition shows the importance of such operators.

Theorem 2.4. *Let $(V, \overline{\partial}_V)$ be a complex vector bundle with a pseudo-connection. Then V has a holomorphic structure whose holomorphic local sections are exactly those sections s satisfying $\overline{\partial}_V s = 0$ if and only if $\overline{\partial}_V^2 = 0$.*

Note that in the case that the manifold M is a surface, the condition of the above theorem is automatically satisfied since a Riemann surface carries no non-zero $(0,2)$ forms. In particular, given any connection ∇ on V, we may define a holomorphic structure via $\nabla^{0,1} = \overline{\partial}_V$.

Now we introduce metrics on vector bundles.

Definition 2.5. *An Hermitian metric on a complex vector bundle V is a smoothly varying family of hermitian forms:*

$$h : V_x \otimes V_x \to \mathbb{C}$$

on each fiber V_x over $x \in M$.

A connection $\nabla \in \mathcal{A}(V)$ is called *unitary* with respect to an Hermitian metric h if for all $s, s' \in \Gamma(V)$,

$$dh(s,s') = h(\nabla s, s') + h(s, \nabla s').$$

Supposing we have a holomorphic vector bundle $(V, \overline{\partial}_V)$ with an Hermitian metric,

Proposition 2.6. *There exists a unique unitary connection ∇ (called the Chern connection) such that $\nabla^{0,1} = \overline{\partial}_V$.*

From here forward, we will fix a Riemann surface structure $\sigma \in \mathcal{T}$ and denote the corresponding Riemann surface $X = (\Sigma, \sigma)$. Given a representation $\rho : \pi \to \mathrm{SL}(n, \mathbb{C})$, form the associated flat vector bundle

$$V_\rho := \widetilde{\Sigma} \times_\rho \mathbb{C}^n,$$

defined as a the quotient of $\widetilde{\Sigma} \times \mathbb{C}^n$ via the diagonal (left) action of π acting on the second factor via composition with the representation ρ. In a flat trivialization, the flat connection is simply the exterior differential acting component-wise on a (local) vector-valued function.

Next, consider a ρ-equivariant map

$$f : \widetilde{\Sigma} \to X_n,$$

where

$$X_n := \{A \in \mathrm{SL}(n, \mathbb{C}) \mid A = A^*, \ \det(A) = 1\}$$

is an explicit model for the symmetric space $\mathrm{SL}(n, \mathbb{C})/\mathrm{SU}(n)$. The left action of $\mathrm{SL}(n, \mathbb{C})$ on X_n is given by

$$g \cdot A = (g^{-1})^* A g^{-1}.$$

Note that such an equivariant map exists since it is equivalent to a section of the fiber bundle

$$\widetilde{\Sigma} \times_\rho X_n$$

which has contractible fibers. Let $\langle \ , \ \rangle$ denote the standard Hermitian inner product on $\mathbb{C}^n$. Given a pair of sections of $s, s' \in \Gamma(V_\rho)$, identify them with equivariant vector-valued maps $\widetilde{\Sigma} \to \mathbb{C}^n$. For $x \in \widetilde{\Sigma}$, f defines a pairing

$$H_f(s, s')(x) = \langle s(x), f(x)s'(x)\rangle.$$

If $\gamma \in \pi$ is acting by deck transformations,

$$\begin{aligned} H_f(s, s')(\gamma x) &= \langle s(\gamma x), f(\gamma x)s'(\gamma x)\rangle \\ &= \langle \rho(\gamma)s(x), (\rho(\gamma)^{-1})^* f(x)\rho(\gamma)^{-1}\rho(\gamma)s'(x)\rangle \\ &= \langle s(x), f(x)s'(x)\rangle \\ &= H_f(s, s')(x). \end{aligned}$$

Thus, H_f defines an Hermitian metric on V_ρ. Conversely, given a metric H on V_ρ, select a positively oriented unitary frame $\{e_i\}$ over the base point $x \in \widetilde{\Sigma}$. Parallel translation using the flat connection gives a global section of the unitary frame bundle which we will also denote $\{e_i\}$. Then define a ρ-equivariant map,

$$\begin{aligned} f : \widetilde{\Sigma} &\longrightarrow X_n \\ y &\longmapsto \{H(e_i(y), e_j(y))\}_{i,j=1,..,n}. \end{aligned}$$

The above two processes are inverse to one another.

Now, denote the flat connection on V_ρ by ∇. Given an Hermitian metric on V_ρ, split the connection (uniquely) as a unitary connection A plus an Hermitian endomorphism $\Psi \in \Omega^1(\mathrm{End}(V_\rho))$,

$$\nabla = A + \Psi.$$

The flatness of ∇ implies

$$0 = F^\nabla = F^A + d^A\Psi + \frac{1}{2}[\Psi, \Psi]. \tag{2.1}$$

The first term is the curvature of the unitary connection A which by unitarity is a 2-form with values in *skew-Hermitian* endomorphisms. The second term is a 2-form with values in Hermitian endomorphisms. Writing $\Psi = \Psi_i^j \ e_j \otimes \phi^i$ where $\{\phi_i\}$ is a dual basis to the $\{e_j\}$ and Φ_i^j are 1-forms on the base, it has as coordinate expression

$$d^A(\Psi) = d\Psi_i^j \ e_j \otimes \phi^i - \Psi_i^j \wedge (A_j^k \ e_k \otimes \phi^i - A_k^i \ e_j \otimes \phi^k).$$

Lastly, the final term combines the wedge product on forms with the Lie bracket (commutator) on endormorphisms;

$$[\Psi, \Psi] := \alpha^i \wedge \alpha^j \otimes [f_i, f_j].$$

The commutator of two Hermitian endomorphisms is *skew-Hermitian*, thus decomposing (2.1) into Hermitian and skew-Hermitian pieces yields a pair of equations:

$$F^A + \frac{1}{2}[\Psi, \Psi] = 0, \tag{2.2}$$

$$d^A \Psi = 0. \tag{2.3}$$

Next, use the complex structure on $X = (\Sigma, \sigma)$ to decompose the covariant derivative d^A and Ψ according to type:

$$d^A = \partial^A + \overline{\partial}^A,$$
$$\Psi = \phi + \phi^{*_H}.$$

Above, $\phi \in \Omega^{1,0}(\mathrm{End}(V_\rho))$, $\phi^{*_H} \in \Omega^{0,1}(\mathrm{End}(V_\rho))$ with the latter adjoint defined using the metric H and the type change given by sending dz to $d\overline{z}$. Since there are no $(0,2)$ nor $(2,0)$ forms on X, (2.2) and (2.3) simplify further,

$$F^A + [\phi, \phi^{*_H}] = 0, \tag{2.4}$$
$$\overline{\partial}^A(\phi) + \partial^A(\phi^{*_H}) = 0.$$

By the discussion about holomorphic structures, the operator $\overline{\partial}^A$ induces a holomorphic structure on V_ρ as well as $\mathrm{End}(V_\rho)$. Wouldn't it be nice if $\overline{\partial}^A \phi = 0$, i.e. ϕ is holomorphic?

Definition 2.7. *The metric H is called harmonic if and only if $\overline{\partial}^A \phi = 0$.*

This definition is bolstered by the following crucial fact:

Proposition 2.8. *H is harmonic if and only if the associated equivariant map*

$$f : \widetilde{X} \to X_n$$

is harmonic.

Remark: Harmonicity is defined with respect to any conformal metric on the Riemann surface $\widetilde{X}$ and the left-invariant Riemannian metric on X_n. Note that we did not define harmonic for non-compact manifolds, nor for equivariant maps: the energy to be minimized here is:

$$\mathcal{E}(f) = \frac{1}{2} \int_D \|df\|^2 dV$$

where $D \subset \widetilde{X}$ is a fundamental domain for the action of π and dV is the volume element of our chosen conformal metric on X.

The analog of the Eells-Sampson theorem in the equivariant case is the following, first proved by Donaldson [Don87] in rank 2, then by Corlette [Cor88] in full generality (see also Labourie [Lab91]).

Theorem 2.9. *Let $\rho \in Hom(\pi, SL(n, \mathbb{C}))$. There exists a ρ-equivariant harmonic map*

$$f : \widetilde{X} \to X_n$$

if and only if the Zariski closure of the image of ρ *is a reductive subgroup of* $SL(n, \mathbb{C})$. *Furthermore, the map is unique up to post-composition by an element which centralizes the image of* ρ.

A reductive subgroup is one which acts completely reducibly, via the adjoint representation, on the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$. The prototypical non-reductive subgroup is the subgroup of upper triangular matrices.

Let $\mathfrak{D}(X)$ denote the space of gauge isomorphism classes of flat vector bundles with harmonic metric (the *De-Rham* moduli space) and $\chi(\Sigma, \mathrm{SL}(n, \mathbb{C}))$ the character variety consisting of conjugacy classes of reductive representations $\pi \to \mathrm{SL}(n, \mathbb{C})$ (the *Betti* moduli space).

The above theorem yields a map, parameterized by a chosen point $\sigma \in \mathcal{T}$,

$$N_\sigma : \chi(\Sigma, \mathrm{SL}(n, \mathbb{C})) \to \mathfrak{D}(X).$$

The above map has an obvious inverse given by taking the holonomy of the flat connection which proves,

Theorem 2.10. *There is a family of isomorphisms* $N_\sigma : \chi(\Sigma, SL(n, \mathbb{C})) \to \mathfrak{D}(X)$ *parameterized by a point* $\sigma \in \mathcal{T}$ *where* $X = (\Sigma, \sigma)$.

Remark: One way to think about this isomorphism is as a section of the bundle over $\chi(\Sigma, \mathrm{SL}(n, \mathbb{C}))$ whose fiber over ρ consists of the (isomorphism classes of) Hermitian metrics on V_ρ. Any smooth section of this bundle yields an identification of the character variety with the space of reductive flat bundles equipped with the metric picked out by the chosen section. This leads to an interesting (albeit vague) question: is there another consistent choice of Hermitian metric on flat bundles which yields a geometrically rich deformation space?

3. Higgs bundles

Finally, we introduce the notion of a Higgs bundle.

Definition 3.1. *A rank-n Higgs bundle over* X *is a triple* $\mathbb{V} = (V, \overline{\partial}_V, \phi)$ *where* $(V, \overline{\partial}_V)$ *is a holomorphic vector bundle and* $\phi \in H^0(X, K \otimes End(V))$.

Remark: $\phi \in H^0(X, K \otimes \mathrm{End}(V))$ says exactly that ϕ is a holomorphic, endomorphism-valued 1-form on X. The tensor ϕ is called the *Higgs* field.

Definition 3.2. *A rank-n Higgs bundle* $\mathbb{V}$ *over a Riemann surface* X *is stable if and only if for every* ϕ*-invariant holomorphic sub-bundle* $W \subset V$,

$$\frac{deg(W)}{rk(W)} < \frac{deg(V)}{rk(V)}.$$

$\mathbb{V}$ *is poly-stable if there exists stable Higgs bundles* $\mathbb{W}_1, ..., \mathbb{W}_k$ *such that*

$$\mathbb{V} = \mathbb{W}_1 \oplus .... \oplus \mathbb{W}_k.$$

The critical Theorem linking Higgs bundles to the story which has unfolded thus far is due to Hitchin [Hit87] (in rank 2) and Simpson [Sim92] in general.

Theorem 3.3. *Let* $\mathbb{V} = (V, \overline{\partial}_V, \phi)$ *be a poly-stable Higgs bundle such that* $det(V)$ *is the trivial holomorphic line bundle. Then there exists a unique (up to unitary* *automorphism) Hermitian metric H on V such that the Chern connection A of H satisfies*

$$F^A + [\phi, \phi^{*_H}] = 0 \tag{3.1}$$

Furthermore, if such a metric exists on any Higgs bundle $\mathbb{V}$, then $\mathbb{V}$ is poly-stable.

Let $\mathfrak{M}_{0,n}(X)$ be the moduli space of degree 0, rank n, poly-stable Higgs bundles with fixed trivial determinant (the *Doulbeaut* moduli space). Here, equivalence is defined up to holomorphic automorphisms commuting with the Higgs field.

Note that we have already seen (3.1) in line (2.4); (3.1) is satisfied if and only if the connection

$$A + \phi + \phi^{*_H}$$

is flat. Furthermore, Theorem 2.9 implies that the holonomy of this flat connection is reductive since the metric H is harmonic; namely $\overline{\partial}^A(\phi) = 0$. This defines a map,

$$\mathfrak{M}_{0,n}(X) \to \chi(\Sigma, \mathrm{SL}(n, \mathbb{C})).$$

Additionally, this map has an inverse since every reductive representation yields a flat bundle with a harmonic metric, which in turn gives rise to a Higgs bundle solving (3.1), thus a poly-stable Higgs bundle. Using suitable topologies (arising from the C^∞-topology on tensors and the topology of pointwise convergence on representations) on the Dolbeaut and Betti moduli space, the following theorem is called the *Non-Abelian Hodge correspondence:*

Theorem 3.4. *The map described above yields a homeomorphism:*

$$\mathfrak{M}_{0,n}(X) \simeq \chi(\Sigma, SL(n, \mathbb{C})).$$

Remark: It is very important to note that this homeomorphism:

(1) Depends on the point $\sigma \in \mathcal{T}$ in a complicated way.
(2) Passes through the transcendental procedure of constructing an equivariant harmonic map.

A very interesting (to the author) future direction is to explore what the (well developed) theory of harmonic maps into symmetric spaces might have to say about the geometric nature of this isomorphism. A basic, but difficult question is the following:

- Can one describe the space of all quasi-Fuchsian representations $\chi(\Sigma, \mathrm{SL}(2, \mathbb{C}))$ purely in terms of the associated Higgs bundles?

The space $\mathfrak{M}_{0,n}(X)$ has many fascinating structures (complex symplectic, hyper-Kähler, quasi-projective variety) which we will not explore here at all. We mention two very important features:

(1) Sending $\phi \to e^{i\theta}\phi$ not only preserves stability, but also preserves the harmonic metric. This action descends to the space $\mathfrak{M}_{0,n}(X)$ and has fixed points which are the critical sub-manifolds for a Morse-Bott function on $\mathfrak{M}_{0,n}(X)$. This allows one, in a number of cases, to compute the rational cohomology of $\mathfrak{M}_{0,n}(X)$ and in particular, the number of connected components. This is one of the greatest success stories involving the theory of Higgs bundles, and is still an active area whose genesis lies in the epic paper of Atiyah and Bott [AB83].

(2) Let $(p_2, ..., p_n)$ be a basis of the conjugation invariant polynomials on $\mathrm{End}(\mathbb{C}^n)$ with $deg(p_i) = i$. Then given a Higgs field ϕ,

$$p_i(\phi) \in H^0(X, K^i)$$

and by the conjugation invariance this descends to a map:

$$F : \mathfrak{M}_{0,n}(X) \to \bigoplus_{i=2}^{n} H^0(X, K^i)$$

called the *Hitchin fibration*. This is a proper map whose generic fibers are Abelian varieties. With respect to a symplectic structure on $\mathfrak{M}_{0,n}(X)$, this is actually a moment map for an algebraically completely integrable Hamiltonian system. Namely, there exists a maximal set of independent Poisson commuting functions whose Hamiltonian vector fields generate flows which lie on the level sets of the Hitchin fibration. Lastly, the $H^0(X, K^2)$ entry is the Hopf differential of the harmonic map associated to that poly-stable Higgs bundle (exercise!).

4. Examples of Higgs bundles and construction of Hitchin component

We begin this section with the Higgs bundle "version" of the first section of these notes. Then we will close with Hitchin's construction of what is now known as the Hitchin component, a component of real representations of π into $\mathrm{SL}(n, \mathbb{R})$ naturally containing the Fricke space of hyperbolic uniformizations of Σ.

As before, we fix a point $\sigma \in \mathcal{T}$ and denote $X = (\Sigma, \sigma)$. Consider the short exact sequence of sheaves

$$1 \to \mathbb{Z}_2 \to \mathcal{O}^* \to \mathcal{O}^* \to 1,$$

where the second arrow takes any locally holomorphic non-vanishing function f to its square f^2. The relevant segment of the long exact sequence in sheaf cohomology is

$$H^1(X, \mathbb{Z}_2) \to H^1(X, \mathcal{O}^*) \to H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z}_2).$$

The arrow

$$w_2 : H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z}_2)$$

computes the mod 2-reduction of the degree (the second *Steifel-Whitney class*) of the line bundle represented by a class in $H^1(X, \mathcal{O}^*)$. Since the canonical bundle of holomorphic 1-forms K has even degree equal to $2g - 2$, it maps under w_2 to zero, hence, by the long exact sequence above, there exists a line bundle $K^{\frac{1}{2}}$ which squares to K. Furthermore, there are

$$|H^1(X, \mathbb{Z}_2)| = |\mathbb{Z}_2^{2g}| = 2^{2g}$$

such inequivalent choices; pick one. Form the holomorphic rank-2 vector bundle

$$V = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}.$$

Then,

$$K \otimes \mathrm{End}(V) = K \oplus K^2 \oplus \mathcal{O} \oplus K$$

whereby if $\alpha \in H^0(X, K^2)$,

$$\phi = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$$

is a well-defined Higgs field. Here, 1 is the constant function, a global section of the sheaf $\mathcal{O}$ of holomorphic functions on X.

This is clearly holomorphic as each entry is holomorphic. Furthermore, there are no non-zero $\phi-$invariant sub-bundles unless $\alpha = 0$ in which case the only invariant sub-bundle is $K^{-\frac{1}{2}}$ which is of negative degree. Hence, (V, ϕ) is a stable Higgs bundle. Additionally, $det(V) = \mathcal{O}$, thus Theorem 3.3 implies that there exists an Hermitian metric on H_α on V with Chern connection A_α such that

$$F^A = -[\phi, \phi^{*_H}]. \tag{4.1}$$

If the metric was not diagonal, then relative to the holomorphic splitting of V an off diagonal entry would appear in the connection form of A_α, but this would imply that the splitting was *not* holomorphic since the connection A is the Chern connection. Thus, the metric has the form

$$H_\alpha = \begin{pmatrix} h_\alpha^{-\frac{1}{2}} & 0 \\ 0 & h_\alpha^{\frac{1}{2}} \end{pmatrix}$$

where h_α is an Hermitian metric on the holomorphic tangent bundle K^{-1}. Now we compute,

$$\phi^{*_H} = H_\alpha^{-1} \overline{\phi}^T H_\alpha = \begin{pmatrix} 0 & h_\alpha \overline{1} \\ h_\alpha^{-1} \overline{\alpha} & 0 \end{pmatrix}.$$

Thus,

$$-[\phi, \phi^{*_H}] = \begin{pmatrix} 1 - h_\alpha^{-2} \alpha \overline{\alpha} & 0 \\ 0 & -1 + h_\alpha^{-2} \alpha \overline{\alpha} \end{pmatrix} h_\alpha dz \wedge d\overline{z}.$$

Note, $h_\alpha^{-2} \alpha \overline{\alpha} = \|\alpha\|_{h_\alpha}^2$ is a scalar-valued function; the norm of α with respect to h_α. The Chern connection takes the form

$$A_\alpha = \begin{pmatrix} \frac{1}{2} a_\alpha^- & 0 \\ 0 & \frac{1}{2} a_\alpha \end{pmatrix}$$

where a_α is the connection 1-form of the metric h_α on K^{-1}. Thus, (4.1) reduces to a single scalar equation:

$$F^{a_\alpha} = -2(1 - \|\alpha\|_{h_\alpha}^2) h_\alpha dz \wedge d\overline{z}.$$

Let's inspect what we have, when $\alpha = 0$ the above equation reads

$$F^{a_\alpha} = -2 h_0 dz \wedge d\overline{z}.$$

This immediately implies that the real part of h_0 furnishes a metric of constant sectional curvature -4 on the surface Σ. Thus, this special case of solving the self-duality equations is equivalent to solving the uniformization theorem. Hitchin [Hit87] showed much more:

Theorem 4.1. *Consider the metric h_α above on K^{-1}. Then the expression,*

$$\hat{h}_\alpha = \alpha dz^2 + (1 + \|\alpha\|_{h_\alpha}^2) h_\alpha dz d\overline{z} + \overline{\alpha} d\overline{z}^2$$

defines a Riemann metric on K^{-1} which has sectional curvature equal to -4. This assignment gives a parameterization of the Fricke space by the space of holomorphic quadratic differentials.

Let us return briefly to the discussion at the beginning of these notes. The following facts are a rewarding exercise for the interested reader:

- The holomorphic quadratic differential α is the Hopf differential of the unique harmonic map isotopic to the identity between from $(\Sigma, \sigma) \to (\Sigma, \hat{h}_\alpha)$.
- The Hitchin map (choose the ad-invariant quadratic polynomial given by minus the determinant) $F : \mathfrak{M}_{0,2}(X) \to H^0(X, K^2)$ takes the Higgs field
$$\phi = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$$
to α. Thus, the Hitchin fibration admits a *section* whose image picks out an entire component of representations lying in the split real form of $\mathrm{SL}(2, \mathbb{C})$.

The moral of this story is that these examples of Higgs bundles are none other than the Harmonic maps parameterization of the Fricke space dressed is slightly fancier language (compare with [Wol89]). The strength of this language is that *it generalizes* and reveals a wealth of structure which was hidden before exploiting the holomorphic geometry.

5. The $\mathrm{SL}(n, \mathbb{R})$ Hitchin component

This section is a condensed presentation of the material in [Hit92].

We now arrive at the goal of these notes, the construction of the Hitchin component for the linear group $\mathrm{SL}(n, \mathbb{R})$. In hindsight, this is a natural generalization of the work in the previous section, and it shows the power of the Higgs bundle theory to reveal new objects, which have turned out to be very geometric.

There is a unique n-dimensional irreducible representation of $\mathrm{SL}(2, \mathbb{C})$ given by the action of $\mathrm{SL}(2, \mathbb{C})$ on homogeneous polynomials of degree $n-1$ in 2 variables. This representation is the $(n-1)$-th symmetric power of the standard 2-dimensional representation. Recall the vector bundle above,
$$V = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}.$$
The $(n-1)$-th symmetric power of this vector bundle is given by
$$S^{n-1}(V) = K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus ... \oplus K^{-\frac{n-3}{2}} \oplus K^{-\frac{n-1}{2}}.$$
The Higgs field becomes
$$\phi = \begin{pmatrix} 0 & (n-1) & 0 & \cdots & & 0 \\ \alpha & 0 & 2(n-2) & \cdots & & 0 \\ 0 & \alpha & 0 & 3(n-3) & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & & & & \ddots & (n-1) \\ 0 & 0 & \cdots & 0 & \alpha & 0 \end{pmatrix}.$$
Now, we have the liberty to do something with the Higgs fields: take $(n-1)$ elements
$$(\alpha_2, \alpha_3, ..., \alpha_n) \in \bigoplus_{i=2}^{n} H^0(X, K^i).$$

Form the new Higgs field, while keeping the holomorphic vector bundle $S^{n-1}(V)$ fixed,

$$\widetilde{\phi} = \begin{pmatrix} 0 & (n-1) & 0 & \cdots & & 0 \\ \alpha_2 & 0 & 2(n-2) & \cdots & & 0 \\ \alpha_3 & \alpha_2 & 0 & 3(n-3) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \alpha_{n-1} & & & & \ddots & (n-1) \\ \alpha_n & \alpha_{n-1} & \cdots & \alpha_3 & \alpha_2 & 0 \end{pmatrix}.$$

A linear algebra calculation (a more clever argument can be found in Hitchin's paper [Hit92]) shows that the Higgs bundle $(S^{n-1}(V), \widetilde{\phi})$ is Higgs stable. Thus, the Theorem of Hitchin and Simpson (Theorem 3.3) guarantees the existence of an Hermitian metric H with Chern connection A such that the self-duality equations are satisfied.

The first question we wish to attack is the following: what are the properties of the holonomy of the flat connection

$$B = A + \widetilde{\phi} + \widetilde{\phi}^{*_H}?$$

For this we will use the following very important Proposition (see [Hit92], [Sim92]).

Proposition 5.1. *Let $\rho \in \chi(\Sigma, SL(n,\mathbb{C}))$ correspond to a Higgs bundle (V,ϕ). (Here the holomorphic structure on V is implicit). Then the Higgs bundle (V^*, ϕ^t) corresponds to the conjugate representation $\overline{\rho}$.*

Thus, fixed points (up to holomorphic automorphism conjugating the Higgs field) of the involution

$$\eta : (V,\phi) \mapsto (V^*, \phi^t)$$

correspond to *real* representations.

Returning to our Higgs bundle $(S^{n-1}(V), \widetilde{\phi})$, the anti-diagonal holomorphic automorphism

$$\begin{pmatrix} 0 & 0 & \cdots & & 1 \\ 0 & \cdots & & 1 & 0 \\ \vdots & & \iddots & & \vdots \\ 0 & 1 & \cdots & & 0 \\ 1 & 0 & \cdots & & 0 \end{pmatrix}$$

maps it to $(S^{n-1}(V)^*, \widetilde{\phi}^t)$. Hence, on the moduli space of Higgs bundles it is fixed by the involution η. Thus, by the previous Proposition 5.1 the holonomy of the flat connection B to which this Higgs bundle corresponds takes values in $\mathrm{SL}(n,\mathbb{R})$.

Next, we wish to show that we have constructed an entire component of real representations: the *Hitchin* component. There is a basis of conjugation invariant polynomials $\{p_2, ..., p_n\}$ such that

$$p_i(\widetilde{\phi}) = \alpha_i.$$

Using these to define the Hitchin fibration, we have constructed a *section*

$$s : \bigoplus_{i=2}^{n} H^0(X, K^i) \to \mathfrak{M}_{0,n}(X).$$

Simply by virtue of being a section, the map s is injective and has closed, connected image. An additional argument (see [Hit92]) shows that s takes values in the smooth part of the moduli space of Higgs bundles. Also, the differential of s is injective. Thus, using the implicit function theorem, the image of s is a closed, connected sub-manifold of $\mathfrak{M}_{0,n}(X)$. Using Corlette's theorem to identity Higgs bundles with reductive representations, we obtain a closed, connected sub-manifold of $\chi(\Sigma, \mathrm{SL}(n,\mathbb{R}))$.

At a smooth point of the character variety $\chi(\Sigma, \mathrm{SL}(n,\mathbb{R}))$, the index theorem can be used to show that its dimension is $|\chi(\Sigma)| \times \dim(\mathrm{SL}(n,\mathbb{R}))$. Meanwhile, a calculation employing the Riemann-Roch theorem yields that the dimension of the Hitchin base is equal to,

$$\begin{aligned}\dim\left(\bigoplus_{i=2}^{n} H^0(X, K^i)\right) &= |\chi(\Sigma)| \sum_{i=1}^{n-1}(2i+1)\\ &= |\chi(\Sigma)|(n^2-1)\\ &= |\chi(\Sigma)|\dim(\mathrm{SL}(n,\mathbb{R})).\end{aligned}$$

Remarkably, this is the same as the dimension of the character variety $\chi(\Sigma, \mathrm{SL}(n,\mathbb{R}))$! Thus, the section s which was previously known to be an immersion is a submersion as well. Applying the inverse function theorem, the image of s is open. As we already know it is closed and connected, this proves that the image of s is a single component of $\chi(\Sigma, \mathrm{SL}(n,\mathbb{R}))$. This is the component which Hitchin identified that is now known as the Hitchin component.

We close with a question: How can we infer geometric properties of the representations in the Hitchin component from differential-geometric properties of the equivariant harmonic map, or equivalently, from the harmonic metric on the associated flat bundle? For example, are there identifiable properties of the harmonic map which guarantee the representation is discrete?

At least for the author, these types of questions are some of the most fascinating surrounding the theory of Higgs bundles. Even in the rank 2 case, the situation is still very murky.

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