MATH 551: HOMEWORK LIST

1. Problems on Riemannian metrics

(1) Consider the smooth map

$$f : \mathbb{D} \to \mathbb{R}^3$$
$$(u, v) \mapsto (u, v, \sqrt{1 - u^2 - v^2})$$

where $\mathbb{D}$ is the open unit disk in $\mathbb{R}^2$. Calculate the Gram matrix of the pullback metric $f^* g_{\mathbb{R}^3}$ with respect to the basis $\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$. Here, $g_{\mathbb{R}^3}$ is the Euclidean metric on $\mathbb{R}^3$. Let (r, θ) denote polar coordinates on an the subset $U \subset \mathbb{D}$ consisting of $(u, v) \in \mathbb{D}$ such that $(u, v) > 0$. Compute the Gram matrix of $f^* g_{\mathbb{R}^3}$ with respect to the basis $\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\}$.

(2) Consider the smooth manifold $\mathbb{H}^2 := \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. Equip $\mathbb{H}^2$ with the Riemannian metric

$$\frac{dx^2 + dy^2}{y^2}.$$

This Riemannian manifold is called the Hyperbolic plane.

- Compute the length of the curve $\gamma(t) = (0, t)$ for $0 < a \leq t \leq b < \infty$.
- Express $(x, y) \in \mathbb{H}^2$ in complex coordinates as $z = x + iy$. Show that the group of invertible linear transformations $SL(2, \mathbb{R})$ with determinant one acts on $\mathbb{H}^2$ via fractional linear transformations:

$$z \mapsto \frac{az + b}{cz + d}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}).$$

- Show that $SL(2, \mathbb{R})$ acts on $\mathbb{H}^2$ by isometries.

(3) (This exercise assumes some exposure to differential forms) Let $\Omega^p(M, \mathbb{R})$ be the space of smooth differential p-forms on a smooth manifold M.

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- Show that a Riemannian metric g on M is equivalent to an isomorphism $\phi_g : \Gamma(TM) \to \Omega^1(M, \mathbb{R})$ such that for all $X, Y \in \Gamma(TM)$,
$$\phi_g(X)(Y) = \phi_g(Y)(X),$$
$$\phi_g(X)(X) \geq 0.$$
Here, $\Gamma(TM)$ is the vector space of smooth vector fields on M.
- Show that the Riemannian metric g induces a smoothly varying inner product on $\Omega^p(M, \mathbb{R})$.

2. Connections and covariant derivative

(1) Consider $\mathbb{R}^n$ with the standard Euclidean metric $g_{\mathbb{R}^n}$.
- Show that, with respect to Euclidean coordinates, all Cristoffel symbols of the Levi-Civita connection vanish.
- In the case of $n = 2$, compute all Christoffel symbols of the Levi-Civita connection of $g_{\mathbb{R}^2}$ with respect to polar coordinates on the upper right quadrant.

(2) Consider the hyperbolic plane $\mathbb{H}^2$ from exercise 1.2. Compute all Cristoffel symbols of $\mathbb{H}^2$ with respect to Euclidean coordinates.

(3) Consider the vector $X_0 := \frac{\partial}{\partial x}$ at the point $(0, 1) \in \mathbb{H}^2$. Compute the parallel transport of X_0 along the curve $\gamma(t) = (0, t)$ where $1 \leq t \leq 20$.

(4) Let (M, g) be a Riemannian manifold and $X, Y \in \Gamma(TM)$ smooth vector fields. Pick $p \in M$ and let $\gamma : [0, 1] \to M$ be a smooth curve in M such that $\gamma(0) = p$ and $\dot{\gamma}(0) = X(p)$. Let $P_{\gamma(t)} : T_pM \to T_{\gamma(t)}M$ be the parallel transport map associated to the Levi-Civita connection. Show that,
$$(\nabla_X Y)(p) = \frac{d}{dt} P^{-1}_{\gamma(t)}(Y(\gamma(t)))|_{t=0}.$$
This is the sense in which the covariant derivative is the infinitesimal form of parallel transport. The covariant derivative measures the first order failure of a vector field to be parallel in a particular direction.

3. Geodesics and completeness

(1) Let (M,g) be a Riemannian manifold. Prove that every $p \in M$ has a strongly convex neighborhood. That is, there exists a neighborhood U of p such that for all $x, y \in \overline{U}$, there exists

a length minimizing geodesic γ joining x and y such that the interior of γ is contained in U.

(2) Let (M, g) be a Riemannian manifold such that the isometry group $\text{Isom}(M, g)$ acts transitively on M. That is, for all $x, y \in M$ there exists an isometry $f \in \text{Isom}(M, g)$ such that $f(x) = y$. Show that (M, g) is a complete Riemannian manifold.

(3) Consider $\mathbb{R}^2$ with the Riemannian metric

$$g = \frac{dx^2 + dy^2}{(1 + x^2 + y^2)^2}$$

Find all the geodesics of this metric which pass through the origin. Can you use this to find all the geodesics? Hint: What are the isometries of g?

(4) Consider the Lie group $SU(2)$ defined as,

$$SU(2) := \{A \in M_2(\mathbb{C}) \mid A\overline{A}^T = I,\ \det(A) = 1\}.$$

- Show that the tangent space at the identity element can be identified with the space of all traceless skew-Hermitian matrices; namely the 2×2 complex matrices of zero trace satisfying $A = -\overline{A}^T$.
- Conclude that the tangent space at $h \in \text{SU}(2)$ can be identified with 2×2 matrices Q such that $h^{-1}Q$ is tangent to the identity. This gives a canonical trivialization $T\text{SU}(2) = SU(2) \times T_e(\text{SU}(2))$.
- Define a pairing on tangent vectors at the identity by $g(A, B) = -trace(AB)$. Show that this defines an inner product on the tangent space at the identity. Use the action of SU(2) on itself by left translation to extend it to a Riemannian metric on all of SU(2).
- A one parameter subgroup of SU(2) is a group homomorphism $\phi : (\mathbb{R}, +) \to \text{SU}(2)$. Show that all geodesics through the origin with respect to the above Riemannian metric are given by one parameter subgroups.

4. Curvature

(1) This problem is a continuation of the problem about SU(2) above. Let $A, B \in T_e(\text{SU}(2))$ be traceless, skew-Hermitian matrices which are orthonormal with respect to the Riemannian metric g. Show that the sectional curvature of the two plane

spanned by A, B is given by

$$\frac{1}{4}\|[A,B]\|^2 = 1.$$

Here, $[A, B]$ is the commutator of matrices. Conclude that $SU(2)$ has constant sectional curvature.

(2) Suppose (M, g) is a 3-dimensional Riemannian manifold and there exists $\lambda \in \mathbb{R}$ such that $Ric(g) = \lambda g$. Prove that g has constant sectional curvature.

(3) Let (M, g) be a Riemannian manifold and $\lambda > 0$. Define a new Riemannian metric by $g_\lambda = \lambda g$. Compute the Cristoffel symbols, Riemann curvature, sectional curvature, Ricci curvature and scalar curvature of g_λ is terms of the associated quantities for g.