PRACTICE FINAL EXAM

1) Consider the 3×3 matrix

$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}.$$

(1) Put A in row reduced echelon form.
(2) For which $b \in \mathbb{R}^3$ does there exist $x \in \mathbb{R}^3$ such that

$$Ax = b.$$

When is this x unique?
(3) What is the rank and nullity of A.
(4) Show that the rows of A are linearly independent. Show that the columns of A are linearly independent.
(5) Show that A is invertible and compute A^{-1}.
(6) Prove without any calculation that the eigenvalues of A are real.
(7) Find all eigenvalues and eigenvectors of A.
(8) Diagonalize A. Use the eigen-decomposition to compute A^{-1}. Compare with your other computation of the inverse.

2) Let $(V, \langle\ ,\ \rangle)$ be an inner product space and let $L : V \to V$ and $K : V \to V$ be linear maps.

(1) Prove that $(L \circ K)^* = K^* \circ L^*$.
(2) Prove that if $v, w \in V$ are orthogonal, then

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

(3) Show that $V \simeq ker(L) \oplus ker(L)^\perp$ where $ker(L)^\perp$ is the orthogonal complement to $ker(L)$.
(4) Show that $Im(L) = ker(L^*)^\perp$.
(5) Suppose $v, w \in V$ are orthogonal. Show that v and w are linearly independent.

3) Consider the vector space V of degree 2 polynomials with real coefficients:

$$V := \{a_0 + a_1 t + a_2 t^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}.$$

(1) Write down two different bases of V. You must verify that what you produce are bases!

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(2) Let $\mathcal{B} := \{1, 1+t, t^2\}$ be an ordered basis of V. Define a map

$$L : V \to V$$
$$a_0 + a_1 t + a_2 t^2 \mapsto a_0 + (a_0 + a_2)t + a_1 t^2$$

Prove that L is a linear map. Write down the matrix representation of L with respect to the basis $\mathcal{B}$.

(3) Verify the rank nullity theorem: $Dim(V) = rk(L) + nullity(L)$.

4) Prove that if A is a non-square matrix, then either the rows or the columns of A are linearly dependent.

5) Suppose V and W are finite dimensional vector spaces. Prove there exists a surjective (onto) linear transformation $T : V \to W$ if and only if $dim(V) \geq dim(W)$.

6) Let $L : V \to V$ be a linear operator on a finite dimensional vector space. Prove that exactly one of the following conditions holds:

(1) The equation $L(v) = b$ has a solution for all vectors $b \in V$.

(2) $nullity(L) > 0$.

7) Let A be an $n \times n$ matrix such that all of the entries of A are integers. If $det(A) = 1$, prove that all of the entries of A^{-1} are also integers.

8) Let A be an $n \times n$ matrix which is diagonalizable. Prove that A^k is diagonalizable for all integers $k > 0$.

9) Let A be an $n \times n$ matrix which is diagonalizable. Prove that the rank of A is equal to the number of non-zero eigenvalues of A.