

Cohomological Obstruction Theory for Brauer Classes and the Period-Index Problem

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Abstract

Let U be a noetherian, quasi-compact, and connected scheme. Let α be a class in $H^2(U_{\text{ét}}, \mathbb{G}_m)$. For each positive integer m , we use the K -theory of α -twisted sheaves to identify obstructions to α being representable by an Azumaya algebra of rank m^2 . We define the spectral index of α , denoted $\text{spi}(\alpha)$, to be the least positive integer such that all of the associated obstructions vanish. Let $\text{per}(\alpha)$ be the order of α in $H^2(U_{\text{ét}}, \mathbb{G}_m)$. We give an upper bound on the spectral index that depends on the étale cohomological dimension of U , the exponents of the stable homotopy groups of spheres, and the exponents of the stable homotopy groups of $B(\mu_{\text{per}(\alpha)})$. As a corollary, we prove that when U is the spectrum of a field of finite cohomological dimension $d = 2c$ or $d = 2c + 1$, then $\text{spi}(\alpha) | \text{per}(\alpha)^c$ whenever $\text{per}(\alpha)$ is not divided by any primes that are small relative to d .

Key Words Brauer groups, twisted sheaves, higher algebraic K -theory, stable homotopy theory.

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1 Introduction

In this paper, we introduce new obstructions for a class $\alpha \in H^2(U_{\text{ét}}, \mathbb{G}_m)$ to be representable by a sheaf of Azumaya algebras of a given rank $[m]^2 \in H^0(U_{\text{ét}}, \mathbb{Z})$. Here, and throughout the paper, U is a noetherian quasi-compact scheme.

As an application of this theory, for a class α in the cohomological Brauer group $H^2(U_{\text{ét}}, \mathbb{G}_m)$ of a scheme U , we introduce a new invariant, $\text{spi}(\alpha)$, which is the least integer $[m] \in H^0(U_{\text{ét}}, \mathbb{Z})$ such that all of the obstructions vanish. We consider the period-index problem for the spectral index $\text{spi}(\alpha)$, and we prove a period-index theorem for $\text{spi}(\alpha)$ when U is the spectrum of a field. Somewhat surprisingly, the exponents of the stable homotopy groups of spheres and of $B\mu_m$ are crucial in the proof of our period-index theorem.

Recall that for $\alpha \in H^2(U_{\text{ét}}, \mathbb{G}_m)$, there are two classical invariants: the period $\text{per}(\alpha)$ which is the order of α in the group $H^2(U_{\text{ét}}, \mathbb{G}_m)$, and the index $\text{ind}(\alpha)$ which

is n if n^2 is the rank of an Azumaya algebra of minimal rank representing α . In general, $\text{per}(\alpha) \mid \text{ind}(\alpha)$. When U is the spectrum of a field k , then the two integers have the same prime divisors. For references on these facts, see the excellent exposition of [13].

Conjecture 1.1 (Period-Index Conjecture). *If k is a field of dimension d , then*

$$\text{ind}(\alpha) \mid (\text{per}(\alpha))^{d-1}.$$

Our new invariant satisfies $\text{spi}(\alpha) \mid \text{ind}(\alpha)$ and $\text{per}(\alpha) \mid \text{spi}(\alpha)$. In some sense, $\text{spi}(\alpha)$ is the cohomological, or homotopical, index. We prove the following theorem.

Theorem 1.2 (Corollary 7.2). *Let k be a field of finite cohomological dimension $d = 2c$ or $d = 2c + 1$. Suppose that $\alpha \in \text{H}^2(k, \mathbb{G}_m)$ has $\text{per}(\alpha) = n$, where $d < 2 \min_{q \mid n}(q) - 1$. Then,*

$$\text{spi}(\alpha) \mid (\text{per}(\alpha))^c.$$

Moreover, in the theorem, we may replace d by the infimum of the q -cohomological dimensions of k for all primes q dividing $\text{per}(\alpha)$.

The spectral index theorem follows from the much more general Theorem 7.1 about the spectral index for classes α on schemes U . This theorem gives a bound for $\text{spi}(\alpha)$ in terms of the étale cohomological dimension d of U , the exponents of the stable homotopy groups of spheres, and the exponents of the stable homotopy groups of $B(\mu_{\text{per}(\alpha)})$.

The dimension of the field k in the Conjecture 1.1 is usually meant to be either the cohomological dimension or d if k is a C_d field. Recall that a field k is said to have property C_d if every homogeneous form $f(x_1, \dots, x_m)$ of degree n has a non-trivial zero if $m > n^d$. See the book of Shatz [27] for the latter notion. In general, there is no obvious known relation between C_d fields and fields of cohomological dimension d . However, C_1 fields have cohomological dimension less than or equal to 1. In [22], the Conjecture 1.1 is attributed to unpublished lecture notes of Colliot-Thélène [7]. Colliot-Thélène suggests the question for function fields of transcendence degree d over algebraically closed fields. The conjecture is known to be true in the following cases:

- k is a p -adic field ($cd(k) = 2$), by class field theory,
- $k(X)$ is a function field of a surface X over an algebraically closed field k ($cd(k(X)) = 2$), due to de Jong [10],
- $k(C)$ is a function field of a curve C over a p -adic field k ($cd(k(C)) = 3$), due to Saltman [26],
- $k(C)$ is a function field of a curve C over a d -local field k ($cd(k(C)) = (d + 1)$) due to Lieblich and Krashen [22], and
- k is a C_2 field and α is a class of period $2^a 3^b$, due to Artin and Harris [2].

To the author's knowledge, no cases are known except those in the papers cited above. As an aside, we mention that the Merkurjev-Suslin theorem [25] says that

$$K_2^M(k)/m \xrightarrow{\cong} \text{H}^2(k, \mu_m^{\otimes 2})$$

when m is invertible in k . The group on the left is the Milnor K -theory group of k , modulo m -divisible elements. The group on the right is isomorphic, after the choice of a primitive m th root of unity, to the m -elementary part of $\text{Br}(k)$. Thus, the Merkurjev-Suslin theorem says that the m -elementary part of the Brauer group is generated by degree m cyclic division algebras. Therefore, if $c > 1$, it follows that we cannot always have $(\text{per}(\alpha))^c | \text{ind}(\alpha)$.

Our obstruction theory uses the theory of α -twisted sheaves, and the associated α -twisted K -theory presheaf of simplicial sets, \mathbf{K}^α on $U_{\text{ét}}$. A necessary condition for α to be represented by an Azumaya algebra of rank $[m]^2$ is that all differentials $d_k^\alpha([m])$ vanish, where the differentials are those from the Brown-Gersten spectral sequence for \mathbf{K}^α :

$$E_2^{s,t} \cong \begin{cases} H^s(U, \pi_{-t}(\mathbf{K}^\alpha)) & \text{if } s+t \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$E_2^{s,t} \simeq H^s(U, \pi_{-t}(\mathbf{K}^\alpha)) \Rightarrow H^{s+t}(U, \mathbf{K}^\alpha),$$

and we identify $H^2(U_{\text{ét}}, \mathbb{Z})$ with $H^2(U_{\text{ét}}, \pi_1(\mathbf{K}^\alpha)) = H^2(U_{\text{ét}}, \mathcal{K}_0^\alpha)$ by Proposition 5.1.

The theory of twisted sheaves has certainly been brought to bear on problems about the Brauer group before; for instance, in [10], [22], and [23]. However, this appears to be the first use of the K -theory of twisted sheaves to analyze Brauer classes.

The notion of using cohomology to create obstructions to the existence of division algebras of specified rank has had success previously in the theory of 2-torsion Brauer classes. For instance, using Hodge theory, Kresch creates in [20] an obstruction class in a quotient of $H^4(X, \mathbb{Z}) \otimes \mathbb{Z}/(2)$. In order for a period 2 Brauer class to be representable by a quaternion algebra, this obstruction class must vanish. Kresch computes this obstruction to establish the existence of rank 16 Azumaya algebras on some smooth projective 3-folds whose restriction to the generic point are biquaternion division algebras. In [8], Colliot-Thélène establishes the result of Kresch without Hodge theory.

In future work, we hope to explore several additional directions:

- First, we would like to eliminate the exclusion of small primes in our result.
- Second, we would like to explore, in terms of the global cohomology theory of presheaves of simplicial sets, the source of the additional $c-1$ or c in the exponent of the classical conjecture. That this is necessary follows from the sharpness of some of the results above. See the end of Section 7 for a more precise description of this problem.
- Third, in the case of cohomological dimension 2 fields, we would like to prove the classical conjecture. In this case, the author shows in [1] that $\text{spi}(\alpha) = \text{per}(\alpha)$.
- Fourth, we would like to explore the possibility of using the cup products

$$\mathbf{K}^\alpha \wedge \mathbf{K}^\beta \rightarrow \mathbf{K}^{\alpha+\beta}$$

in conjunction with the Merkurjev-Suslin theorem to study the period-index problem. For instance, suppose that every element of $K_2^M(k)/p$ can be written as

a sum of at most $\lambda_p(k)$ symbols, where p is invertible in k . Then, every element $\alpha \in \text{Br}(k)$ with $\text{per}(\alpha) = p$ is equal (in the Brauer group) to the product of at most $\lambda_p(k)$ cyclic algebras. So, we have

$$\text{ind}(\alpha) | p^{\lambda_p(k)}.$$

As the period and index coincide for cyclic algebras, there may be significant additional information that can be brought to bear on the study of α using the cup-product above. The number $\lambda_p(k)$ is studied, for instance, in [21], [19], and [4]. There is a conjecture slightly stronger than the period-index conjecture for C_2 and cohomological dimension 2 fields. Namely, it is the conjecture that every division algebra is cyclic for these fields. This would imply that, for such a field k , we have $\lambda_p(k) = 1$ for all primes p invertible in k .

- Fifth, we would like to explore the relation between the obstruction of Kresch and our Theorem 6.1.

Now, we describe the contents of the paper. In Section 2, we describe the sheaf and stack-theoretic machinery which underlies our approach to the Brauer group. The fundamental notion is that of twisting the gluing data of a stack via a 2-cocycle in some sheaf.

This is used in Section 3 to create stacks of twisted sheaves \mathbf{Proj}^α , as in [9]. The K -theory presheaves \mathbf{K}^α are then the point-wise applications of the K -theory functor on symmetric monoidal categories whose morphisms are isomorphisms (henceforth, symmetric monoidal groupoids). Then, we demonstrate an important application of the twisting to create twisted stacks of sheaves of faithful μ_n -sets. For a class $\beta \in H^2(U_{\text{ét}}, \mu_n)$ that goes to a class $\alpha \in H^2(U_{\text{ét}}, \mathbf{G}_m)$ under the natural map, we get a stack \mathbf{nSets}^β and a morphism of stacks of symmetric monoidal groupoids $\mathbf{nSets}^\beta \rightarrow \mathbf{Proj}^\alpha$. In Section 3.3, we compute some of the exponents of the stalks of the homotopy sheaves of $\mathbf{K}(\mathbf{nSets}^\beta)$. This data is the key input for our proof of the period-spectral index theorem.

In Section 4, we recall the formalism of presheaves of simplicial sets and the associated Brown-Gersten spectral sequences. None of this material is new, except, perhaps, a description of fibrant $K(G, 1)$ -spaces for non-abelian sheaves of groups G .

In Section 5, we prove the important fact that the sheaves of abelian groups $\pi_i \mathbf{K}^\alpha$ and $\pi_i \mathbf{K}$ are isomorphic for all $i \geq 0$. The same proof shows that the sheaves $\pi_i \mathbf{K}(\mathbf{nSets}^\beta)$ and $\pi_i \mathbf{K}(\mathbf{nSets})$ are isomorphic.

Finally, in Section 6, we establish the obstruction theorem, showing that in order for α to be represented by an Azumaya of rank $[m]^2$ it is necessary for $d_k^\alpha([m]) = 0$ for all $k \geq 2$ in the Brown-Gersten spectral sequence for \mathbf{K}^α .

The final section, Section 7, contains the proof of the period-spectral index theorem.

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2 Sheaves

The purpose of this section is to introduce the primary objects of study below, namely Azumaya algebras and stacks of twisted sheaves. An excellent source for much of this material is the thesis of Căldăraru [9], although of course it goes back to the work of Grothendieck and Giraud on non-abelian cohomology [15].

Throughout, C will denote a locally ringed Grothendieck site, and U will be an object of C such that $C \downarrow U$ has enough points. We assume that C is closed under finite fiber products, and therefore that the topology of the site C is given by a pre-topology, in the sense of [3, Definition II.1.3]. This just means that we can describe everything with covering families, instead of necessarily using sieves.

2.1 Stacks

In order to be precise in our definitions later, we must fix notation for stacks over a site with terminal object $C \downarrow U$. For us, a stack will be a fibered category over $C \downarrow U$ that satisfies descent and has fixed clivage.

Let $F : T \rightarrow C$ be a functor. For objects V of C , we will denote by T_V the category consisting of those objects A of T such that $F(A) = V$. The morphisms of T_V are the morphisms a of T such that $F(a) = id_V$.

Definition 2.1. *A morphism $f : A \rightarrow B$ in T is called cartesian if, for every morphism $g : A' \rightarrow B$ such that $F(g) = F(f)$, there exists a unique $h : A' \rightarrow A$ such that $g = f \circ h$. In this case, we call A the pull-back of B under $F(f) : F(A) \rightarrow F(B)$, and we call f a pull-back morphism.*

Definition 2.2. *The category $F : T \rightarrow C$ is called pre-fibered if, for every morphism $\phi : V \rightarrow W$ in C and every object B in T_W , there is a cartesian morphism $f : A \rightarrow B$ such that $F(f) = \phi$. Of course, this implies that A is an object of T_V . The category $F : T \rightarrow C$ is called fibered if it is pre-fibered and if the composition of cartesian morphisms is cartesian.*

Definition 2.3. *A choice of a cartesian pull-back morphism $f_\phi^B : A_\phi^B \rightarrow B$ for every $\phi : V \rightarrow W$ and B in T_W is called a clivage for F .*

A clivage for the fibered category $F : T \rightarrow C$ is exactly what is required to define pull-back functors on the fibers. Indeed, for $\phi : V \rightarrow W$ in C , the clivage defines a unique map $\phi^* : T_W \rightarrow T_V$ on objects given by taking the domain of the pull-back maps: $B \mapsto A_\phi^B$. Given a morphism $b : B' \rightarrow B$ in T_W , then $F(b \circ f_\phi^{B'}) = F(f_\phi^B)$. By definition of cartesian morphisms, there is a unique morphism $\phi^*(b) : A_\phi^{B'} \rightarrow A_\phi^B$. Given $B \xrightarrow{b} B' \xrightarrow{c} B''$, the composition $\phi^*(c) \circ \phi^*(b)$ satisfies the cartesian lifting

property for the maps $c \circ b \circ f_\phi^B : A_\phi^B \rightarrow B''$ and $A_\phi^{B''} \rightarrow B''$. Thus, ϕ^* preserves composition and is a functor.

For each chain of morphisms $U \xrightarrow{\pi} V \xrightarrow{\phi} W$, there is a natural transformation $\lambda_{\pi, \phi} : \pi^* \circ \phi^* \Rightarrow (\phi \circ \pi)^*$ such that the following diagrams of natural transformations commutes for every $T \xrightarrow{\theta} U \xrightarrow{\pi} V \xrightarrow{\phi} W$:

$$\begin{array}{ccc} \theta^* \circ \pi^* \circ \phi^* & \xrightarrow{\theta^* \circ \lambda_{\pi, \phi}} & \theta^* \circ (\phi \circ \pi)^* \\ \lambda_{\theta, \pi \circ \phi^*} \downarrow & & \lambda_{\theta, \phi \circ \pi} \downarrow \\ (\pi \circ \theta)^* \circ \phi^* & \xrightarrow{\lambda_{\pi \circ \theta, \phi}} & (\phi \circ \pi \circ \theta)^* \end{array}$$

This is established in a similar way as the existence of the clivage pull-back functors.

Now, we suppose that the base category C has the structure of a Grothendieck site, and we let $F : T \rightarrow C$ be a fibered category with clivage. Then, given a covering $\phi : \mathcal{V}_I \rightarrow W$ in C , we define a descent category $D = \underline{Des}(\phi : \mathcal{V}_I \rightarrow W)$. The cover is made up of morphisms $\phi_i : V_i \rightarrow W$ for $i \in I$. Let $p_1 : V_i \times_U V_j \rightarrow V_i$ and $p_2 : V_i \times_U V_j \rightarrow V_j$ for any i, j . Let $p_{12} : V_i \times_U V_j \times_U V_k \rightarrow V_i \times_U V_j$. Define p_{13} and p_{23} similarly. Then, for any $i, j, k \in I$, we have equalities of morphisms in C

$$\begin{aligned} p_1 \circ p_{13} &= p_1 \circ p_{12} \\ p_2 \circ p_{12} &= p_1 \circ p_{23} \\ p_2 \circ p_{13} &= p_2 \circ p_{23}, \end{aligned}$$

An object of the descent category D consists of an object A_i of T_{V_i} and isomorphisms $a_{ij} : p_2^*(A_j) \rightarrow p_1^*(A_i)$ such that

$$\begin{aligned} p_{13}^*(p_2^*(A_k)) &\xrightarrow{\lambda} (p_2 \circ p_{13})^*(A_k) = (p_2 \circ p_{23})^*(A_k) \xrightarrow{\lambda^{-1}} p_{23}^*(p_2^*(A_k)) \xrightarrow{p_{23}^*(a_{jk})} \\ p_{23}^*(p_1^*(A_j)) &\xrightarrow{\lambda} (p_1 \circ p_{23})^*(A_j) = (p_2 \circ p_{12})^*(A_j) \xrightarrow{\lambda^{-1}} p_{12}^*(p_2^*(A_j)) \xrightarrow{p_{12}^*(a_{ij})} \\ p_{12}^*(p_1^*(A_i)) &\xrightarrow{\lambda} (p_1 \circ p_{12})^*(A_i) = (p_1 \circ p_{13})^*(A_i) \xrightarrow{\lambda^{-1}} p_{13}^*(p_1^*(A_i)) \end{aligned}$$

agrees with the morphism

$$p_{13}^*(p_2^*(A_k)) \xrightarrow{p_{13}^*(a_{ik})} p_{13}^*(p_1^*(A_i)).$$

A clivage is called a scindage in the case that all the natural transformations λ are the identity transformation. In this case, composition of pull-back functors is strict:

$$\pi^* \circ \phi^* = (\phi \circ \pi)^*.$$

In a stack where this is the case, the above maps simplify greatly, and we require the more familiar formula

$$p_{12}^*(a_{ij}) \circ p_{23}^*(a_{jk}) = p_{13}^*(a_{ik}),$$

or even more simply just

$$a_{ij} \circ a_{jk} = a_{ik}$$

on $V_{ijk} = V_i \times_U V_j \times_U V_k$.

Let $A_I = (A_i, a_{ij})$ and $B_I = (B_i, b_{ij})$ be two objects of D . Then, a morphism $A_I \rightarrow B_I$ consists of morphisms $c_i : A_i \rightarrow B_i$ such that the squares

$$\begin{array}{ccc} p_2^*(A_j) & \xrightarrow{a_{ij}} & p_1^*(A_i) \\ p_2^*(c_j) \downarrow & & p_1^*(c_i) \downarrow \\ p_2^*(B_j) & \xrightarrow{b_{ij}} & p_1^*(B_i) \end{array}$$

are commutative.

Note that there is a natural functor $d : T_W \rightarrow \underline{Des}(\phi : \mathcal{V}_I \rightarrow W)$. For an object A of T_W , we let the objects of $d(A)$ be $\phi_i^*(A)$. The morphisms a_{ij} are

$$p_2^*(\phi_j^*(A)) \xrightarrow{\lambda_{\phi, p_1}} (p_2 \circ \phi_j)^*(A) = (p_1 \circ \phi_i)^*(A) \xrightarrow{(\lambda_{\phi_i, p_1})^{-1}} p_1^*(\phi_i^*(A)).$$

For a morphism $c : A \rightarrow B$ of T_W , we let $c_i = \phi_i^*(c)$. Then, one checks easily that the c_i determine a morphism $d(A) \rightarrow d(B)$ in the descent category.

Definition 2.4. *A stack over a Grothendieck site C is a fibered category $F : T \rightarrow C$ with clivage such that the functors $T_W \rightarrow \underline{Des}(\phi : V \rightarrow W)$ are equivalences of categories.*

For details, please see [17, Exposé VI].

A morphism of stacks $T \rightarrow T'$ is a morphism of C -categories that respects the clivage of both stacks. Thus, it is a functor $G : T \rightarrow T'$ such that $F' \circ G = F$. The functor G induces functors $G_V : T_V \rightarrow T'_V$ for all V in C . The respect of clivage means that for all $\phi : V \rightarrow W$ in C , the diagram

$$\begin{array}{ccc} T_W & \xrightarrow{\phi^*} & T_V \\ G_W \downarrow & & G_V \downarrow \\ T'_W & \xrightarrow{\phi^*} & T'_V \end{array}$$

is commutative.

Unlike in stacks themselves, the restriction of stacks is well-defined without choices. If $F : T \rightarrow C \downarrow U$ is a stack, and if $\phi : V \rightarrow U$ is a morphism in $C \downarrow U$, then we may define the stack $\phi^*(T) \rightarrow C \downarrow V$ as being the sub-category of T consisting of objects A with $F(A)$ in $C \downarrow V$ and morphisms a with $F(a)$ in $C \downarrow V$. Thus, $\phi^*(T)$ is the pull-back in the category of categories over $C \downarrow U$. Note that, using this definition, we have equalities $\pi^*(\phi^*(T)) = (\phi \circ \pi)^*(T)$ whenever $\pi : W \rightarrow V$ and $\phi : V \rightarrow U$.

2.2 Gluing stacks

Essentially by definition, one can glue stacks. It will be worthwhile to detail concretely how this is done. Let $C \downarrow U$ be a Grothendieck site with a terminal object U . If $V \rightarrow U$ is an object of $C \downarrow U$, then we will let $C \downarrow V$ denote the induced site with terminal object V . Suppose that $F_i : T_i \rightarrow C \downarrow V_i$ are stacks for a cover $\phi : \mathcal{V}_I \rightarrow U$. In order to descend to a stack on to $C \downarrow U$, we must first give equivalences of stacks $\sigma_{ij} : p_2^*(T_j) \rightarrow p_1^*(T_i)$, for all $i, j \in I$, where the p_i are the natural projections. We should also require natural transformations

$$\gamma_{ijk} : p_{23}^*(\sigma_{jk}) \circ p_{12}^*(\sigma_{ij}) \Rightarrow p_{13}^*(\sigma_{ik}),$$

for all $i, j, k \in I$ to fill in the (non-commutative) square

$$\begin{array}{ccccc} p_{23}^*(p_2^*(T_k)) & \xrightarrow{p_{23}^*(\sigma_{jk})} & p_{23}^*(p_1^*(T_j)) & \xrightarrow{=} & p_{12}^*(p_2^*(T_j)) \\ \downarrow = & & & & \downarrow p_{12}^*(\sigma_{ij}) \\ & & & & p_{12}^*(p_1^*(T_i)) \\ & & & & \downarrow = \\ p_{13}^*(p_2^*(T_k)) & \xrightarrow{p_{13}^*(\sigma_{ik})} & p_{13}^*(p_1^*(T_i)) & & \end{array}$$

where $p_{ij} : V \times_U V \times_U V \rightarrow V \times_U V$ are the natural projections. Finally, we require that γ satisfy a cocycle condition: we require that the two natural transformations

$$p_{34}^*(\sigma) \circ p_{23}^*(\sigma) \circ p_{12}^*(\sigma) \Rightarrow p_{24}^*(\sigma) \circ p_{12}^*(\sigma) \Rightarrow p_{14}^*(\sigma)$$

and

$$p_{34}^*(\sigma) \circ p_{23}^*(\sigma) \circ p_{12}^*(\sigma) \Rightarrow p_{34}^*(\sigma) \circ p_{13}^*(\sigma) \Rightarrow p_{14}^*(\sigma)$$

over $V \times_U V \times_U V \times_U V$ agree.

Now, for any object of $C \downarrow U$ given by $\phi : W \rightarrow U$, we define a descent category $D = \underline{Des}(W \times_U \mathcal{V}_I \rightarrow W)$. The idea is then that these descent categories define the stack globally on $C \downarrow U$. An object of D consists of objects A_i of $T_{W \times_U V_i}$ for all $i \in I$, together with isomorphisms

$$\beta_{ij} : \sigma_{ij}(p_2^*(A_j)) \rightarrow p_1^*(A_i),$$

such that the diagram

$$\begin{array}{ccc} p_{12}^*(\sigma_{ij})(p_{23}^*(\sigma_{jk})(p_3^*(A_k))) & \xrightarrow{p_{12}^*(\sigma_{jk})(p_{23}^*(\beta_{jk}))} & p_{12}^*(\sigma_{ij})(p_2^*(A_j)) \\ \gamma_{ijk} \downarrow & & \downarrow p_{12}^*(\beta_{ij}) \\ p_{13}^*(\sigma_{ik})(p_3^*(A_k)) & \xrightarrow{p_{13}^*(\beta_{ik})} & p_1^*(A_i) \end{array}$$

is commutative, where the p_i morphisms are the natural projections from $V_i \times_U V_j \times_U V_k$. Note that we omit the natural transformations λ from the stacks T_i . This is only a matter of convenience. We leave to the reader the definition of morphisms in the descent categories D and morphisms across fibers.

Proposition 2.5. *The category whose objects are descent data as defined above for all objects $\phi : W \rightarrow U$ in $C \downarrow U$ defines a stack over U .*

2.3 Gerbes and the Cohomological Brauer Group

If A is a sheaf of groups on a site C , then we define a stack of A -torsors $Tors(A)$. The fiber $Tors(A)_V$ consists of $A|_V$ -torsors on V . A map of A -torsors $a : A \rightarrow B$ that lies over a morphism $\phi : V \rightarrow W$ is an isomorphism $A \xrightarrow{\cong} \phi^*(B)$. We will write \mathbf{Pic} for the stack of \mathbb{G}_m -torsors. In fact, these torsor stacks are gerbes.

Definition 2.6. *A gerbe over a Grothendieck site $C \downarrow U$ is a stack G satisfying three conditions: the fiber categories must all be groupoids; there is some cover $\mathcal{V}_I \rightarrow U$ such that each G_{V_i} is non-empty; for two objects $A, B \in G_W$, there is a cover $\phi : \mathcal{V}_I \rightarrow W$ such that there are isomorphisms $\phi_i^*(A) \xrightarrow{\cong} \phi_i^*(B)$ in each G_{V_i} .*

This definition may be summed up by saying that a gerbe is a stack whose fibers are groupoids such that the stalks are connected.

Definition 2.7. *Let A be a sheaf of abelian groups on $C \downarrow U$. Any gerbe G locally equivalent to $Tors(A)$ is called an A -gerbe. Here, local equivalence means that there is a covering morphism $\phi : \mathcal{V}_I \rightarrow U$, and there are equivalences of stacks $\phi_i^*(G) \rightarrow \phi_i^*(Tors(A))$ for all i .*

It is standard knowledge that equivalence classes of A -gerbes are classified by the cohomology group $H^2(U_{\acute{e}t}, A)$, when A is a sheaf over a scheme U in the étale topology. We will not prove this here, but we will indicate how to go from an A -gerbe to a cocycle, and vice-versa.

To say that a gerbe G is an A -gerbe is to say that there is a cover \mathcal{V}_I of U , there are objects $a_i \in G_{V_i}$, and there exist isomorphisms $\sigma_i : \text{Aut}(a_i) \xrightarrow{\cong} A|_{V_i}$. Indeed, in this case, if $b \in G_{V_i}$, then $\text{Iso}(a_i, b)$ is a $\text{Aut}(a_i)$ -torsor, and hence, via σ_i^{-1} , a $A|_{V_i}$ -torsor. Together, the a_i and σ_i give an equivalence of gerbes $G|_{V_i} \rightarrow Tors(A)|_{V_i}$. Showing that it is actually an equivalence simply amounts to using descent. Indeed, if $\text{Iso}(a_i, b)$ is the trivial A -torsor, then there is an isomorphism $a_i \rightarrow b$ over V_i . On the other hand, if L is an A -torsor over V_i , then we can take a cover on which it is trivial, and use the gluing datum to create a descent data for a_i . Then, we get an object b_L of G_{V_i} with $\text{Iso}(a_i, b_L)$ isomorphic to L .

Recall how to associate an element of $\check{H}^2(U, A)$ to an A -gerbe G . Let \mathcal{V}_I as above be a cover of U that trivializes G . Let, for each $i, j \in I$, \mathcal{W}_{ij} be a cover of $V_{ij} = V_i \times_U V_j$ such that on each W_{ij}^l there is a morphism $\theta_{ij}^l : a_i|_{W_{ij}^l} \rightarrow a_j|_{W_{ij}^l}$. Set $Z_{ijk}^{lmn} = W_{ij}^l \times_U W_{ik}^m \times_U W_{jk}^n$. Then,

$$\sigma_i((\theta_{ik}^m)^{-1}|_{W_{ijk}^{lmn}} \circ \theta_{jk}^n|_{W_{ijk}^{lmn}} \circ \theta_{ij}^l|_{W_{ijk}^{lmn}})$$

gives an element of $A(Z_{ijk}^{lmn})$. It is not hard to check that this gives us a 2-cocycle. And, the cocycle in $\check{H}^2(U, A)$ is well-defined and depends only the gerbe G up to equivalence of stacks.

Now, we come for the first time to a construction which will be fundamental for the entire work. It is the idea that a class $\alpha \in \check{H}^2(U, A)$ tells us exactly how to twist the gerbe $Tors(A)$ to get a gerbe $Tors(A)^\alpha$ whose associated cohomology class is α . The basic construction will be repeated to obtain the stacks of twisted sheaves and the twisted stacks of finite μ_n -sets.

Fix $\alpha \in \check{H}^2(U, A)$. Let α be determined by a class $\alpha_{ijk} \in \check{H}^2(\mathcal{V}_I, A)$. Then, on each V_i in \mathcal{V}_I , we let $G_i = Tors(A)|_{V_i}$. On the overlap $V_i \times_U V_j$, we let

$$\sigma_{ij} = Id : p_2^*(G_j) \xrightarrow{=} p_1^*(G_i).$$

Thus, the overlap maps are all the identity. What we twist are the natural transformations γ_{ijk} . We let γ_{ijk} be multiplication by α_{ijk} , as a natural transformation of the identity on the category of A -torsors. The cocycle condition for γ_{ijk} follows from the cocycle condition for α_{ijk} . The corresponding gerbe determined by this gluing data is called $Tors(A)^\alpha$. The key point is that we can do a similar construction for any stack on which A acts canonically.

2.4 Twisted Coherent Sheaves

In this spirit, fix $\alpha \in H^2(U_{\acute{e}t}, \mathbb{G}_m)$. Suppose that α is represented by $\alpha_{ijk} \in \check{H}^2(\mathcal{V}_I, \mathbb{G}_m)$. On each open set V_i we set $T_i = \mathbf{Proj}|_{V_i}$, where \mathbf{Proj} is the stack of finite rank projective modules. Using the exact same method of twisting, where we let α_{ijk} transform the identity by multiplication, we obtain the stack of α -twisted finite rank projective modules \mathbf{Proj}^α .

This is a somewhat more belabored definition of twisted sheaves than is usual, so we use the descent categories defined above to recapture the more traditional definition. To give an object of \mathbf{Proj}_W^α , we give objects P_i of $\mathbf{Proj}_{W \times_U V_i}$. We must give isomorphisms

$$\beta_{ij} : p_2^*(A_j) \rightarrow p_1^*(A_i),$$

over $V_i \times_U V_j$, recalling that the functors σ_{ij} are identity functors. Finally, the β_{ij} must make the squares

$$\begin{array}{ccc} p_3^*(A_k) & \xrightarrow{p_{23}^*(\beta_{jk})} & p_2^*(A_j) \\ \gamma_{ijk} \downarrow & & p_{12}^*(\beta_{ij}) \downarrow \\ p_3^*(A_k) & \xrightarrow{p_{13}^*(\beta_{ik})} & p_1^*(A_i) \end{array}$$

commutative. This recalls the precise definition of α -twisted sheaves, for instance as it appears in [9].

In this case, if we consider \mathbf{Proj} as a stack of symmetric monoidal categories under \oplus , then the natural transformations of the identity γ_{ijk} are in fact symmetric monoidal transformations of the identity functors. Therefore, the stacks \mathbf{Proj}^α possess a natural structure of stacks of symmetric monoidal categories.

2.5 Azumaya Algebras

It is not obvious at first whether there should in general exist non-trivial global α -twisted locally free and finite rank sheaves in \mathbf{Proj}_U^α for $\alpha \in H^2(U_{\text{ét}}, \mathbb{G}_m)$. In fact, this is equivalent to the question of whether α is representable by an Azumaya algebra \mathcal{A} . Indeed, given a non-trivial α -twisted finite rank projective sheaf \mathcal{E} , the endomorphism sheaf $\text{End}(\mathcal{E})$ is an Azumaya algebra representing α . In the other direction, this follows from the fact that \mathcal{A} is locally a matrix algebra over \mathcal{O}_U . For details, we again refer to [9].

3 K-Theory

We take as K -theory functor the level one part of a functor from symmetric monoidal categories to E_∞ -spectra. See [28, Appendix A], and the references there. The level zero will not do, as the version of the Brown-Gersten spectral sequence for presheaves of simplicial sets does not see differentials emerging from $H^0(U, \pi_0 X)$. Therefore, if T is a symmetric monoidal groupoid, then $\pi_k(\mathbf{K}(T)) = K_{k-1}(T)$ for $k \geq 1$.

3.1 Twisted K-Theory

Let $\alpha \in H^2(U_{\text{ét}}, \mathbb{G}_m)$. We associate to α a twisted K -theory presheaf \mathbf{K}^α by setting

$$\mathbf{K}^\alpha(V) = \mathbf{K}(\mathbf{Proj}_V^\alpha),$$

where K -theory is that of symmetric monoidal categories.

3.2 K-Theory of Monomial Matrices

Now, let $\alpha \in H^2(U_{\text{ét}}, \mu_n)$. Let \mathbf{nSets} denote the stack of sheaves of finite and faithful μ_n -sets on U . This stack becomes a stack of symmetric monoidal categories under the disjoint sum operation of μ_n sets. Because μ_n is abelian, given $\theta \in \Gamma(V, \mu_n)$ and a $\mu_{n,V}$ -set A , we get an isomorphism of μ_n sets $\theta_* : A \rightarrow A$ where θ_* acts as multiplication by θ . This isomorphism is compatible with the monoidal structure on \mathbf{nSets} , so θ_* acts as a natural symmetric monoidal transformation of the identity of $\mathbf{nSets}|_V$. As above, we can therefore construct a new symmetric monoidal stack \mathbf{nSets}^α by gluing using a 2-cocycle representative for α .

Proposition 3.1. *Let $\beta \mapsto i_*(\beta)$ in the canonical map $i_* : H^2(U_{\text{ét}}, \mu_n) \rightarrow H^2(U_{\text{ét}}, \mathbb{G}_m)$ induced by $i : \mu_n \rightarrow \mathbb{G}_m$. Then, there is a natural map S of symmetric monoidal stacks $\mathbf{nSets}^\beta \rightarrow \mathbf{Proj}^{i(\beta)}$ such that, if A is μ_n -torsor, with class $[A] \in H^1(U_{\text{ét}}, \mu_n)$, then $S(A)$ is a \mathbb{G}_m -torsor with class $i_*([A])$ via the map $i_* : H^1(U_{\text{ét}}, \mu_n) \rightarrow H^1(U_{\text{ét}}, \mathbb{G}_m)$.*

Proof. Indeed, we see that if $\mathcal{V}_I \rightarrow U$ is a cover over which β is trivial, then, on each open set V_i of the cover, we have a natural map

$$S_i : \mathbf{nSets}|_{V_i} \rightarrow \mathbf{Proj}|_{V_i},$$

which, on μ_n -torsors, is extension of scalars to \mathbb{G}_m followed by the map from \mathbb{G}_m -torsors to line bundles and sends disjoint unions of μ_n -torsors to direct sums of line bundles. The natural transformations that β and $i_*(\beta)$ induce on the triple intersections are compatible with the S_i maps. Therefore, they glue together to give the desired map. \square

For $\alpha \in H^2(U_{\acute{e}t}, \mu_n)$, we will let \mathbf{T}^α denote the presheaf

$$\mathbf{T}^\alpha(V) = \mathbf{K}(\mathbf{nSets}^\alpha).$$

Every μ_n -set is a disjoint union of μ_n -torsors. The stalk of the stack \mathbf{nSets} at a geometric point $\bar{x} \rightarrow U$ is therefore equivalent to

$$\coprod_k S_k \wr \mu_n(k(\bar{x})),$$

where S_k is the symmetric group on k letters, and $S_k \wr \mu_n$ is the wreath product. This is true in the étale topology because the local ring of a geometric point is Henselian. By the Barratt-Priddy-Quillen-Segal theorem [29, Lemma 2.5], the K -theory space of this category is weak equivalent to $(B\mu_n(k(\bar{x})))_+$. Stably, this space is equivalent to $B\mu_n(k(\bar{x})) \vee S^0$. Therefore, the stable homotopy is

$$K_k(\mathbf{nSets}_{\bar{x}}) \xrightarrow{\cong} \pi_k^s((B\mu_n(k(\bar{x})))_+) \xrightarrow{\cong} \pi_k^s(B\mu_n(k(\bar{x}))) \oplus \pi_k^s,$$

where $\pi_k^s = \pi_k^s(S^0)$.

Henceforth, we will let $n_{\bar{x}}$ be the order of $\mu_n(k(\bar{x}))$. If n is prime to the characteristic of $k(\bar{x})$, then $n_{\bar{x}} = n$. Otherwise, if $k(\bar{x})$ is characteristic p , and if $v_p(n)$ denotes the p -adic valuation of n at p , then $n_{\bar{x}} = n/p^{v_p(n)}$. Then, $\mu_n(k(\bar{x})) \xrightarrow{\cong} \mathbb{Z}/(n_{\bar{x}})$.

The classifying space $B\mu_n(k(\bar{x}))$ splits up as

$$B\mu_n(k(\bar{x})) \xrightarrow{\cong} \bigvee_{q|n} B\mu_{q^{v_q(n)}}(k(\bar{x})) \xrightarrow{\cong} \bigoplus_{q|n_{\bar{x}}} \mathbb{Z}/(q^{v_q(n_{\bar{x}})}).$$

3.3 Stable Homotopy of Classifying Spaces

I thank Peter Bousfield for telling me about the next two propositions, which are known to experts.

Proposition 3.2. *Let $G = \mathbb{Z}/(p^g)$. Then, for $0 < k < 2p - 2$, the stable homotopy group $\pi_k^s(BG)$ is isomorphic to $\mathbb{Z}/(p^g)$ for k odd and 0 for k even. If,*

$$G = \mathbb{Z}/(n) = \bigoplus_{q|n} \mathbb{Z}/(q^{v_q(n)}),$$

then for $0 < k < 2 \min_{q|n}(q) - 2$, we have $\pi_k^s(BG) \cong G$.

Proposition 3.3. *Let $\pi_k^s(p)$ denote the p -primary component of π_k^s . Then, $\pi_k^s(p) \subseteq \pi_k^s(B\mathbb{Z}/(p))$.*

Proposition 3.4. *Let $0 < k < 2p - 3$. Then, the p -primary component $\pi_k^s(p)$ of π_k^s is zero. And,*

$$\pi_{2p-3}^s(p) \subseteq \pi_{2p-3}^s(B\mathbb{Z}/(p)) \cong \mathbb{Z}/(p).$$

Proof. The first statement follows from the description $\pi_k^s(p) = \mathbb{Z}/(p)$ for $k = 2l(p - 1) - 1$ for $l = 1, \dots, (p - 1)$, and $\pi_k^s(p) = 0$ for the other k satisfying $0 < k < 2p(p - 1) - 2$. See [12]. The second statement follows from Propositions 3.2 and 3.3. \square

Corollary 3.5. *Denote by m_k the exponent of π_k^s for $k \geq 1$. If $G = \mathbb{Z}/(n) = \bigoplus_{q|n} \mathbb{Z}/(q^{v_q(n)})$, then, for*

$$0 < j < 2 \min_{q|n}(q) - 2,$$

$H^k(U_{\text{ét}}, \pi_j(\mathbf{T}))$ is annihilated by $n \cdot m_j$ when j is odd and by m_j when j is even.

4 Descent Spectral Sequence

4.1 Closed Model Structure on Simplicial Presheaves

Let C be a Grothendieck site. We will denote by $\mathbf{Pre}(C)$ and $\mathbf{Shv}(C)$ the categories of presheaves and sheaves on C , and we will write $\mathbf{sPre}(C)$ and $\mathbf{sShv}(C)$ for the categories of simplicial presheaves and simplicial sheaves. For a presheaf X in either category of presheaves, we will denote by $X \rightarrow \tilde{X}$ its sheafification.

We use the following closed model category structure on simplicial presheaves. The cofibrations are the pointwise cofibrations. Thus, $X \rightarrow Y$ is a cofibration if and only if $X(U) \rightarrow Y(U)$ is a monomorphism for every object U of C . We denote this by $X \hookrightarrow Y$. For an object U of C , there is a site with terminal object $C \downarrow U$. Each presheaf or sheaf on C restricts to a presheaf or sheaf on $C \downarrow U$. For a simplicial presheaf X , an object U of C , and a basepoint $x \in X(U)_0$, we get presheaves of homotopy groups $\pi_k^p(X|U, x)$:

$$(f : V \rightarrow U) \mapsto \pi_k(|X(V)|, f^*(x)),$$

where $|X(V)|$ denotes the geometric realization of the simplicial set $X(V)$. We will denote the associated homotopy sheaves by $\pi_k(X|U, x)$. We call $w : X \rightarrow Y$ a weak equivalence if it induces an isomorphism of homotopy sheaves

$$\pi_k(X|U, x) \xrightarrow{\cong} \pi_k(Y|U, w(x))$$

for all choices of U , all basepoints x of $X(U)$, and all $k \geq 0$. Local weak equivalence (or, the weak equivalences of any model category) are denoted by $X \xrightarrow{\sim} Y$. The fibrations are all maps having the right lifting property with respect to all acyclic cofibrations. A fibration is denoted $X \twoheadrightarrow Y$. That this is a simplicial closed model category is proven in [18]. We will refer to these classes of morphisms more specifically as global fibrations, global cofibrations, and local weak equivalences.

In [11], Dugger, Hollander, and Isaksen describe the globally fibrant objects in more familiar sheaf-theoretic language. Namely, if X is a simplicial presheaf, and if $\mathcal{V} \rightarrow U$

is a hypercover, then we let $X_{\mathcal{V}}$ denote the cosimplicial space associated to \mathcal{V} . There is a canonical augmentation $X(U) \rightarrow X_{\mathcal{V}}$. The result is that X is globally fibrant if and only if $X(U) \rightarrow X_{\mathcal{V}}$ is a weak equivalence for all hypercovers $\mathcal{V} \rightarrow U$. There is an analogous description of general fibrations.

There are other types of morphisms we use, namely pointwise weak equivalences and pointwise fibrations. A pointwise weak equivalence is a morphism $f : X \rightarrow Y$ such that $X(U) \xrightarrow{\sim} Y(U)$ is a weak equivalence of simplicial sets for all objects U of \mathcal{C} . Two pointwise weak equivalent sheaves are local weak equivalent, and two local weak equivalent fibrant presheaves are pointwise weak equivalent. A pointwise fibration is a morphism $f : X \rightarrow Y$ such that every $f : X(U) \rightarrow Y(U)$ is a fibration of simplicial sets. We will say that X is pointwise Kan or pointwise fibrant if $X \rightarrow *$ is a pointwise fibration

Note that if a simplicial presheaf is pointed, then the homotopy presheaves and sheaves above may be defined globally.

Let F be a functor from simplicial sets to simplicial sets such that $F(\emptyset) = \emptyset$, or from pointed simplicial sets to pointed simplicial sets such that $F(*) = *$. If X is a simplicial presheaf, our convention will be to denote by FX the pointwise application of F to X , so that $(FX)(U) = F(X(U))$ for all U . For instance, below, $\text{cosk}_n X$ will be the pointwise n -coskeleton of X . If F preserves weak equivalences of simplicial sets, then it preserves local weak equivalences of simplicial presheaves. This is the case, for instance, for the coskeleta functors and for the Ex functor. In particular, we can always replace X with the local weak equivalent $\text{Ex}^{\infty} X$, which is pointwise Kan.

4.2 Presheaves and Sheaves of Eilenberg-Mac Lane Type

The ideas of this section, with the exception of the $\mathbf{K}(\tilde{G}, 1)$ -spaces for non-abelian \tilde{G} , are due to Brown and Gersten [6].

Let A be a presheaf of abelian groups, groups, or pointed sets, and let n be a non-negative integer, with $n = 0, 1$ if A is a presheaf of groups, or $n = 0$ if A is a presheaf of pointed sets. We say that a presheaf X of pointed simplicial sets is a $K^p(A, n)$ -space if $\pi_n^p(X) \cong A$ and $\pi_m^p(X) = *$ for $m \neq n$. We will say that X is a $K(\tilde{A}, n)$ -space, where \tilde{A} is the sheafification of A , if $\pi_n(X) \cong \tilde{A}$ and $\pi_m(X) = *$ for $m \neq n$. Of course, the condition on presheaves is much stronger. We will often consider a $K^p(A, n)$ -space or $K(\tilde{A}, n)$ -space to include the information of a specific isomorphism $\pi_n^p X \xrightarrow{\cong} A$ or $\pi_n X \xrightarrow{\cong} \tilde{A}$.

For each Eilenberg-Mac Lane type (\tilde{A}, n) , we define a fixed $K(\tilde{A}, n)$, which we show to be fibrant. Let \tilde{S} be a sheaf of pointed sets. Define $\mathbf{K}(\tilde{S}, 0)$ to be the constant pointed simplicial set $\tilde{S}(U)$ on each object U . Evidently, $\mathbf{K}(\tilde{S}, 0)$ is a $K(\tilde{S}, 0)$ space. It is not difficult to check directly that $\mathbf{K}(\tilde{S}, 0) \rightarrow *$ satisfies the right lifting property with respect to all acyclic cofibrations. Indeed, if $A \hookrightarrow B$ is a cofibration which is also a local weak equivalence, and if we are given a map $f : A \rightarrow \mathbf{K}(\tilde{S}, 0)$, then we proceed as follows. Note that it suffices, by the definition of $\mathbf{K}(\tilde{S}, 0)$ to define $B(U) \rightarrow \mathbf{K}(\tilde{S}, 0)(U)$ on $B(U)_0$, where the subscript 0 denotes the level 0 of the simplicial set $B(U)$. Let $x \in B(U)_0$. Then, there is a covering sieve $R \subseteq \text{Hom}(-, U)$ such that for each $\phi : V \rightarrow U$ in R , the element $\phi^*(x)$ is homotopic to an element

$y_V \in A(V)_0$. Thus, we get elements $f(y_V) \in \tilde{S}(V)$. By definition, on $V_0 \times_U V_1$, $f(y_{V_0}) = f(y_{V_1})$. Therefore, since \tilde{S} is a sheaf, the $f(y_V)$ glue together to give a unique element of $\tilde{S}(U)$. We let this be $f(x)$. This is a well-defined extension. It follows that $\mathbf{K}(\tilde{S}, 0)$ is a fibrant presheaf (actually, a sheaf) of simplicial sets.

Now, let X be a $K(\pi_0 X, 0)$ -space. We show that there is a canonical map $X \rightarrow \mathbf{K}(\pi_0 X, 0)$. Indeed, this is the composition $X \rightarrow \mathbf{K}^p(\pi_0^p X, 0) \rightarrow \mathbf{K}(\pi_0 X, 0)$ given by sheafification, where $\mathbf{K}^p(\pi_0^p X, 0)(U)$ is the constant simplicial set $\pi_0^p X(U)$. Evidently, this is a local weak equivalence, so that $\mathbf{K}(\pi_0 X, 0)$ is a canonical fibrant resolution for X .

We turn to the definition of $\mathbf{K}(\tilde{G}, 1)$ when \tilde{G} is a sheaf of groups. Denote by $Tors(\tilde{G})$ the stack of \tilde{G} -torsors on U . Define

$$\mathbf{K}(\tilde{G}, 1)(V) = B(Tors(\tilde{G})_V),$$

where $B(Tors(\tilde{G})_V)$ is the classifying space of the category $Tors(\tilde{G})_V$.

To show that $\mathbf{K}(\tilde{G}, 1)$ is fibrant is not difficult. Indeed, it follows that as \tilde{G} is a sheaf and $Tors(\tilde{G})$ is a stack that we can argue as we did for the proof of the fibrancy of the $\mathbf{K}(\tilde{S}, 0)$ spaces. Indeed, given an acyclic cofibration $A \hookrightarrow B$, and a map $A \rightarrow \mathbf{K}(\tilde{G}, 1)$, this works to define the map from $B_{\leq 1}$ to $\mathbf{K}(\tilde{G}, 1)_{\leq 1}$, on 1-skeletons. But, as a classifying space of a category is determined by its 1-skeleton this map extends to all of B .

We can always replace a $K(\tilde{S}, n)$ presheaf X by an $(n-1)$ -reduced $K(\tilde{S}, n)$ presheaf. An $(n-1)$ -reduced presheaf is one which has exactly one simplex in each dimension less than n on every object U of C . Indeed, by applying the Ex^∞ functor, we may assume that X is pointwise fibrant. Then $\text{cosk}_n X \rightarrow \text{cosk}_{n-1} X$ are pointwise fibrations. The fiber F is a $K(\tilde{S}, n)$ space and, by construction, is $(n-1)$ -reduced. Moreover, X and F are canonically locally weak equivalent to $\text{cosk}_n X$, so that X and F are canonically isomorphic in $\text{Ho}(\mathbf{sPre}(C))$.

We want to show that any $K(\pi_1(X), 1)$ -space X is naturally isomorphic to $\mathbf{K}(\pi_1(X), 1)$ in the homotopy category $\text{Ho}(\mathbf{sPre}(C))$. As above we may assume that X is 0-reduced. Then, there is a canonical map $X \rightarrow \mathbf{K}(\pi_1(X), 1)$ given by sending the point in X_0 to the trivial $\pi_1(X)$ -torsor, and by sending loops in X to the corresponding loops in $\pi_1(X)$.

Finally, let \tilde{A} be a sheaf of abelian groups. Let $\tilde{A} \rightarrow I$ be an injective resolution. Denote by \mathbf{K} the functor from non-negatively graded chain complexes of abelian groups to simplicial abelian sheaves. Let I be the chain complex with $I_n = I^{-n}$. Then, we let $\mathbf{K}(\tilde{A}, n) = \mathbf{K}(\tau_{\leq 0} I.[n])$, where τ is the good truncation. There is a canonical map

$$\mathbf{K}(\tilde{A}[n]) \rightarrow \mathbf{K}(\tilde{A}, n),$$

which is a local weak equivalence by construction. It is shown in [14, lemma 2] that $\mathbf{K}(\tilde{A}, n)$ is fibrant. Indeed, Gillet and Soulé show that it is fibrant as a sheaf of simplicial abelian groups, so that it is clearly fibrant as a presheaf of simplicial sets, since any diagram in $\mathbf{sPre}(C)$

$$\begin{array}{ccc} A & \longrightarrow & \mathbf{K}(\tilde{A}, n) \\ \downarrow & & \downarrow \\ B & \longrightarrow & * \end{array}$$

may be factored through the sheafification of A and B to get a diagram in $\mathbf{sShv}(C)$

$$\begin{array}{ccccc} A & \longrightarrow & \tilde{A} & \longrightarrow & \mathbf{K}(\tilde{A}, n) \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & \tilde{B} & \longrightarrow & * \end{array}$$

The fibrancy for $\mathbf{K}(\tilde{A}, n)$ results since sheafification preserves cofibrations.

We want to show that any $K(\tilde{A}, n)$ space X is naturally isomorphic to $\mathbf{K}(\tilde{A}, n)$ in $\mathrm{Ho}(\mathbf{sPre}(C))$. Using the coskeleta argument as above, we may reduce to the case that X is $(n-1)$ -reduced. Let $X \xrightarrow{\sim} \tilde{X}$ be the sheafification, and let $\mathbb{Z}\tilde{X}$ denote the free abelian simplicial sheaf associated to X . Since X is $(n-1)$ -reduced, $N\mathbb{Z}\tilde{X}$ is zero in degrees less than n , where N denotes the normalized chain complex functor. The map to homology and the Hurewicz theorem [24, theorem 13.6] give a natural quasi-isomorphism of chain complexes of sheaves $N\mathbb{Z}\tilde{X} \rightarrow A[n]$. Therefore, by the Dold-Kan correspondence, there is a natural local weak equivalence $X \xrightarrow{\sim} \tilde{X} \xrightarrow{\sim} \mathbf{K}(\tilde{A}[n])$. Composing with the canonical local weak equivalence $\mathbf{K}(\tilde{A}[n]) \xrightarrow{\sim} \mathbf{K}(\tau_{\leq 0}I.[n])$, we have shown that X is weak equivalent to $\mathbf{K}(\tilde{A}, n)$.

We have proved the following proposition.

Proposition 4.1. *Let X be a $K(\pi_n X, n)$ -space. If $n = 0$, then there is a canonical (in $\mathbf{sPre}(C)$) local weak equivalence $X \rightarrow \mathbf{K}(\pi_0 X, 0)$. If $n > 0$, there is a canonical (in $\mathrm{Ho}(\mathbf{sPre}(C))$) isomorphism $X \rightarrow \mathbf{K}(\pi_n X, n)$. The spaces $\mathbf{K}(\tilde{A}, n)$, when \tilde{A} is a sheaf of abelian groups, are uniquely defined up to unique isomorphisms in $\mathrm{Ho}(\mathbf{sPre}(C))$. The spaces $\mathbf{K}(\tilde{S}, 0)$ and $\mathbf{K}(\tilde{G}, 1)$ are uniquely defined.*

4.3 Hypercohomology

Given a pointed simplicial presheaf Y , we will denote by $\mathbb{H}Y$ a fibrant resolution of Y . For an object U of C , we define the hypercohomology groups of U with coefficients in Y as

$$\mathbb{H}^{-n}(U, Y) = \pi_n \Gamma(U, \mathbb{H}Y).$$

A different choice of fibrant resolution yields the same hypercohomology along with a unique isomorphism between the hypercohomology groups. Using Postnikov towers, we may create a local to global spectral sequence that relates the cohomology groups of homotopy sheaves to hypercohomology groups. Let

$$Y(n) = \mathbb{H} \mathrm{cosk}_n \mathbb{H}Y$$

for $n \geq 0$. Set $Y(-1) = *$. We can and do choose these in such a way that each $Y(n) \rightarrow Y(n-1)$ is a global fibration for all $n \geq 0$. One knows that global fibrations are pointwise fibrations. Therefore, $\Gamma(U, Y(n)) \rightarrow \Gamma(U, Y(n-1))$ is a tower of fibrations of fibrant simplicial sets. The spectral sequence is the spectral sequence of this tower, as described in [5, section IX.4], [16, section XII.6], or [28, paragraph 5.42]. The fiber of $Y(n) \rightarrow Y(n-1)$ is a fibrant resolution of the $K^p(\pi_n^p \mathbb{H}Y, n)$ -space $F(n)$, the

point-wise fiber of $\text{cosk}_n \mathbb{H}Y \rightarrow \text{cosk}_{n-1} \mathbb{H}Y$. Therefore, it is itself a $K(\pi_n \mathbb{H}Y, n)$ -space. But, by the definition of local weak equivalences, $\pi_n \mathbb{H}Y \simeq \pi_n Y$. We will denote this fiber by $\mathbb{H}F(n)$.

Before going further, we must determine the hypercohomology of a fibrant $K(\tilde{S}, n)$ -space. We know from Proposition 4.1 that any such space has the same hypercohomology as our standard $\mathbf{K}(\tilde{S}, n)$ -spaces. If $n = 0$, we have

$$\mathbb{H}^k(U, \mathbf{K}(\tilde{S}, 0)) = \begin{cases} \Gamma(U, \tilde{S}) & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

If $\tilde{S} = \tilde{A}$, and abelian sheaf, then we have from our construction above

$$\mathbb{H}^k(U, \mathbf{K}(\tilde{A}, n)) = \Gamma(U, \mathcal{H}_{-k}(\tau_{\leq 0} I.[n])) = \begin{cases} \mathbb{H}^{n+k}(U, A) & \text{if } -n \leq k \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Finally, in the case that $\tilde{S} = \tilde{G}$ and $n = 1$, we have by construction that

$$\pi_0 \Gamma(U, \mathbf{K}(\tilde{G}, 1)) = \mathbb{H}^1(U, \tilde{G}),$$

the group of \tilde{G} -torsors. The base-point of $\mathbf{K}(\tilde{G}, 1)$ is the trivial \tilde{G} -torsor \tilde{G} . Thus, we can consider $\pi_1 \Gamma(U, \mathbf{K}(\tilde{G}, 1))$ as its group of \tilde{G} -automorphisms. This corresponds to picking a base-point over U , and hence it is isomorphic to $\mathbb{H}^0(U, \tilde{G})$. There are no non-trivial higher homotopy groups. Therefore, the table above in the case of an abelian sheaf, holds as well for a non-abelian sheaf of groups.

We use the traditional (re)indexing for this spectral sequence. Therefore, in the language of exact triples,

$$\begin{aligned} D_2^{s,t} &= \pi_{-s-t} \Gamma(U, Y(-t)), \\ E_2^{s,t} &= \pi_{-s-t} \Gamma(U, \mathbb{H}F(-t)). \end{aligned}$$

The differential is $d_2 : E_2^{s,t} \rightarrow E_2^{s+2,t-1}$. We see that the E_2 -terms are

$$E_2^{s,t} \cong \begin{cases} \mathbb{H}^s(U, \pi_{-t} Y) & \text{if } s+t \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is important to note that by the construction of $\mathbf{K}(I.[n])$ and by lemma 4.1 the identifications of the E_2 -terms as sheaf cohomology are in fact functorial in both U and X . We will write

$$E_2^{s,t} \simeq \mathbb{H}^s(U, \pi_{-t} X) \Rightarrow \mathbb{H}^{s+t}(U, X),$$

although this should not be taken to mean that there is the usual sort of convergence, or that the convergence are to the groups on the right. We will mention this a little more below, but for the most part this does not affect the arguments of this paper.

5 Homotopy Sheaves are Isomorphic

Proposition 5.1. *Fix an element $\alpha \in \mathbb{H}^2(U_{\text{ét}}, \mathbb{G}_m)$. Then, the homotopy sheaves $\pi_n(\mathbf{K}^\alpha)$ and $\pi_n(\mathbf{K})$ are naturally isomorphic. Similarly, if $\beta \in \mathbb{H}^2(U_{\text{ét}}, \mu_n)$, then $\pi_n(\mathbf{T}^\beta) \cong \pi_n(\mathbf{T})$.*

Proof. We include a proof for the case of $\alpha \in \mathbb{H}^2(U_{\text{ét}}, \mathbb{G}_m)$. The proof of the other case is identical.

Let $\mathcal{U}_I \rightarrow U$ be a cover over which α is trivial.

Then, the gerbe \mathbf{Pic}^α is trivial on \mathcal{U}_I . Thus, there exist α -twisted line bundles \mathcal{L}_i on each U_i . These define equivalences $\theta_i : \mathbf{Proj}|_{U_i} \rightarrow \mathbf{Proj}^\alpha|_{U_i}$ for all i given by

$$\theta_i(V)(\mathcal{P}) = \mathcal{L}_i \otimes \mathcal{P},$$

when $V \rightarrow U_i$. These equivalences induce point-wise weak equivalences of K -theory presheaves: $\theta_i : \mathbf{K}|_{U_i} \rightarrow \mathbf{K}^\alpha|_{U_i}$. It follows that on U_i there are isomorphisms of homotopy presheaves:

$$\theta_i : \pi_n^p(\mathbf{K})|_{U_i} \rightarrow \pi_n^p(\mathbf{K}^\alpha)|_{U_i}.$$

We show that the θ_i glue at the level of homotopy sheaves. Since in our cover we might have $U_i = U_j$, and we can take different line bundles \mathcal{L}_i and \mathcal{L}_j , this will imply that the resulting morphisms on homotopy sheaves of K -theory are independent of the choice of the line bundles \mathcal{L}_i . It will also show that the morphisms do not depend on the cover \mathcal{U}_I .

It suffices to check that, on $U_{ij} = U_i \times_U U_j$, the autoequivalence on $\mathbf{Proj}|_{U_{ij}}$ given by tensoring by $\mathcal{M}_{ij} = \mathcal{L}_i^{-1} \otimes \mathcal{L}_j$ is locally homotopic to the identity. But, we can take a trivialization of \mathcal{M}_{ij} , over a cover \mathcal{V} of U_{ij} . So, on each element V of \mathcal{V} , there is an isomorphism $\sigma_V : \mathcal{O}_{U_V} \rightarrow \mathcal{M}_{ij}|_V$. This induces a natural transformation from the identity to $\theta_i^{-1} \circ \theta_j$ on V . So, on V , we see that $\theta_i|_V = \theta_j|_V : \pi_n^p(\mathbf{K})|_V \rightarrow \pi_n^p(\mathbf{K}^\alpha)|_V$. It follows that the θ_i glue to give isomorphisms of sheaves

$$\theta : \pi_n(\mathbf{K}) \rightarrow \pi_n(\mathbf{K}^\alpha).$$

□

Corollary 5.2. *Denote by m_k the exponent of π_k^s for $k \geq 1$. If*

$$G = \mathbb{Z}/(n) = \bigoplus_{q|n} \mathbb{Z}/(q^{v_q(n)}),$$

and if $\alpha \in \mathbb{H}^2(U_{\text{ét}}, \mu_n)$, then, for $0 < j < 2 \min_{q|n}(q) - 2$, $\mathbb{H}^k(U_{\text{ét}}, \pi_j(\mathbf{T}^\alpha))$ is annihilated by $n \cdot m_j$ when j is odd and by m_j when j is even.

6 Obstruction Theory

6.1 Obstruction Theory

Let X be a simplicial presheaf. We define two subgroups (subsets if $t = 0$) of $\mathbb{H}^0(U, \pi_t X)$. First, the global reduced subgroup is defined as

$$\mathbb{H}_{\text{red}}^0(U, \pi_t X) = \text{im}(\pi_t \Gamma(U, X) \rightarrow \mathbb{H}^0(U, \pi_t X)).$$

Second, the liftable subgroup is defined as

$$H_{\text{lift}}^0(U, \pi_t X) = \text{im}(\pi_t G \rightarrow H^0(U, \pi_t X)),$$

where G is the inverse limit of the U -sections of the Postnikov tower for X , and the map is induced by $G \rightarrow \Gamma(U, X(t))$ and sheafification:

$$\pi_t G \rightarrow \pi_t \Gamma(U, X(t)) \rightarrow \Gamma(U, \pi_t X(t)) \cong \Gamma(U, \pi_t X).$$

The commutative diagram

$$\begin{array}{ccccc} \pi_t \Gamma(U, X) & \longrightarrow & \pi_t \Gamma(U, \mathbb{H}X) & \longrightarrow & \pi_t G \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma(U, \pi_t X) & \xlongequal{\quad} & \Gamma(U, \pi_t X) & \xlongequal{\quad} & \Gamma(U, \pi_t X) \end{array}$$

shows that $H_{\text{red}}^0(U, \pi_t X) \subseteq H_{\text{lift}}^0(U, \pi_t X)$. Therefore, a necessary condition for an element of $H^0(U, \pi_t X)$ to lie in $H_{\text{red}}^0(U, \pi_t X)$ is for it to be annihilated by all differentials. For $t = 0$, this condition is trivial, since $d_k = 0$ on $H^0(U, \pi_0 X)$ for $k \geq 2$. For $t > 0$, $d_j : E_j^{0,-t} \rightarrow E_j^{0+j,-t-j+1}$, and $j - t - j + 1 \leq 0$ if and only if $-t + 1 \leq 0$. Therefore, we can use the spectral sequence for an obstruction theory for $\pi_t X$ when $t > 0$.

6.2 Obstruction Theory for Cohomological Brauer Classes

Now, we apply the last section to cohomological Brauer classes.

Theorem 6.1. *Let $\alpha \in H^2(U_{\text{ét}}, \mathbf{G}_m)$, where U is a noetherian quasi-compact scheme. Fix a class $[m] \in H^0(U, \mathbf{Z})$. A necessary condition for α to be represented by an Azumaya algebra of rank $[m]^2$ is that $d_k^\alpha([m]) = 0$ for all $k \geq 2$, where the differentials d_k^α are those of the Brown-Gersten spectral sequence for \mathbf{K}^α . If, for some $[m]$ with $n|[m]$, we have that $d_k([m])$ is non-torsion, then α is not in the image of the Brauer group.*

Proof. Suppose that α is represented by an Azumaya algebra \mathcal{A} . Then, there exists an α -twisted locally free and finite rank sheaf \mathcal{E} that is defined on all of U and such that $\mathcal{A} \cong \text{End}(\mathcal{E})$. In particular, if \mathcal{A} is of rank $[m]^2$, then \mathcal{E} is of rank $[m]$. Therefore, there is a rank $[m]$ element in $\pi_1^p \mathbf{K}^\alpha(U)$. This maps to $[m]$ in $H^0(U_{\text{ét}}, \pi_1 \mathbf{K}^\alpha)$, which we see, by Proposition 5.1, is isomorphic to $H^0(U_{\text{ét}}, \mathbf{Z})$. Therefore, by Section 6.1, $[m]$ lies in $H_{\text{red}}^0(U_{\text{ét}}, \pi_1 \mathbf{K}^\alpha)$, and hence in $H_{\text{lift}}^0(U_{\text{ét}}, \pi_1 \mathbf{K}^\alpha)$. It follows that

$$d_k^\alpha([m]) = 0$$

for $k \geq 2$ in the Brown-Gersten spectral sequence for \mathbf{K}^α . This completes the proof. \square

7 The Period-Index Problem

In this section, we apply the methods developed above to the period-index problem.

Let $\alpha \in H^2(U_{\acute{e}t}, \mathbf{G}_m)$, where U is of finite étale cohomological dimension. Then, there is a unique smallest positive integer $spi(\alpha)$ such that

$$d_k^\alpha([spi(\alpha)]) = 0$$

for all $k \geq 2$, where we take the differentials in the Brown-Gersten spectral sequence for \mathbf{K}^α . We call this the spectral index. By the obstruction theory, it is the smallest integer that *might* be the rank of an α -twisted locally free finite rank sheaf. Evidently,

$$per(\alpha) | spi(\alpha) | ind(\alpha).$$

We introduce some notation before the next theorem. Denote by m_j the exponent of π_j^s , the j th stable homotopy group of S^0 , and let n_j^α denote the exponent of $\pi_j^s(B\mu_{per(\alpha)})$. Finally, let l_j^α denote the exponent of $\pi_j^s \oplus \pi_j^s(B\mathbf{Z}/(per(\alpha)))$. So, l_j^α is the least common multiple of m_j and n_j^α .

Theorem 7.1. *Let U be a noetherian quasi-compact scheme such that the étale cohomological dimension of U with coefficients in finite sheaves is a finite integer d . Let $\alpha \in H^2(U_{\acute{e}t}, \mathbf{G}_m)$. Then,*

$$spi(\alpha) | \prod_{j=1}^{d-1} l_j^\alpha.$$

Proof. Let β be a lift of α to $H^2(U_{\acute{e}t}, \mu_{per(\alpha)})$. We will let d_k^β denote the i th differential in the Brown-Gersten spectral sequence for \mathbf{T}^β . As the class $[1]$ in $H^0(U_{\acute{e}t}, \pi_1(\mathbf{T}^\beta))$ maps to the class $[1]$ in $H^0(U_{\acute{e}t}, \pi_1(\mathbf{K}^\alpha))$, if $d_k^\beta([m]) = 0$, then $d_k^\alpha([m]) = 0$. The differential d_k^β lands in a subquotient of $H^k(U, \pi_{k-1}(\mathbf{T}^\beta))$. But, we have seen that the stalks of \mathbf{T}^β are stably weak homotopy equivalent to $B\mu_{per(\alpha)}(k(\bar{x})) \wedge S^0$. Therefore, d_k^β lands in a group of exponent at most l_{k-1}^α . As the differentials d_k^β all vanish for $k > d$, the theorem follows. \square

Corollary 7.2. *Let k be a field of finite cohomological dimension $d = 2c$ or $d = 2c + 1$. Suppose that $\alpha \in H^2(k, \mathbf{G}_m)$ has $per(\alpha) = n$, where $d < 2 \min_{q|n}(q) - 1$. Then,*

$$spi(\alpha) | (per(\alpha))^c.$$

Proof. Under the numerical condition $d < 2 \min_{q|n}(q) - 1$, the hypotheses of Corollary 5.2 hold. Therefore, for $1 < j < d$, we have $n_j = 1$ for j even, and $n_j = n$ for j odd. For $0 < j < d$, Proposition 3.4 says that for all primes q dividing n , the exponent of q in l_j is determined by the exponent of q in n_j . Let

$$m_j^* = m_j / \prod_{q|n} q^{v_q(m_j)}.$$

So m_j^* is the part of m_j free of all primes that divide $\text{per}(\alpha)$. It follows that

$$\text{spi}(\alpha) \mid \left((\text{per}(\alpha))^c \cdot \prod_{i=1}^{d-1} m_j^* \right).$$

On the other hand, as k is a field, the primes divisors of $\text{per}(\alpha)$ and $\text{spi}(\alpha)$ are the same. So,

$$\text{spi}(\alpha) \mid (\text{per}(\alpha))^f$$

for some positive integer f . Now, as $H_{\text{lift}}^0(U_{\text{ét}}, \pi_1(\mathbf{K}^\alpha))$ is cyclic, it follows that

$$\text{spi}(\alpha) \mid (\text{per}(\alpha))^{\min(c,f)} \mid (\text{per}(\alpha))^c.$$

This completes the proof. □

Corollary 7.3. *By the proof of the theorem and corollary, we may replace d by the n -torsion cohomological dimension d_n of k in the statement of Corollary 7.2.*

Proof. Indeed, if k is of n -torsion cohomological dimension d_n , and if G is a finite sheaf, then $H^g(k, G)$ has no n -primary component, for $g > d_n$. □

The condition $2c < 2 \min_{q|n}(q) - 1$ excludes no primes for curves, the prime 2 for surfaces, and the primes 2, 3 for three-folds and four-folds. At these primes, further results may be obtained by consulting tables of stable homotopy groups.

We would have liked to conjecture that for $\alpha \in \text{Br}(k)$, we had $\text{spi}(\alpha) = \text{ind}(\alpha)$. However, the sharpness results of Saltman [26] in the case of curves over p -adic fields and Krashen [22] in the case of a class of fields of every cohomological dimension show that this cannot be the case in general.

The new period-index problem (index-index problem), to determine the relation between $\text{spi}(\alpha)$ and $\text{ind}(\alpha)$ splits naturally into two problems. The first is to determine if, in the language of Section 6.1, the class $\text{spi}(\alpha)$ lifts to a class of $\pi_0(G)$. This follows in our case from general results on the convergence of Brown-Gersten spectral sequences under finiteness hypotheses. The second problem is to compute $\pi_0(\mathbf{K}^\alpha(k)) \rightarrow \pi_0(G)$ is an isomorphism. Very little appears to be known about how to approach this sort of problem.

We now discuss two further questions in this direction. We may define the integer $n\text{spi}(\alpha)$ as the least integer such that $d_k^3([n\text{spi}(\alpha)]) = 0$ for all $k \geq 2$. As a preliminary, we would like to know that this is independent of the lift of α to $H^2(k, \mu_{\text{per}(\alpha)})$. The arguments above show that when k is a field of p -cohomological dimension $d = 2c$ or $d = 2c + 1$, then $n\text{spi}(\alpha) \mid n^c$. The first question is whether this is always an equality. The second question is whether, in this case, $n\text{spi}(\alpha) = \text{spi}(\alpha)$. We hope in a future work to consider both questions.

In another direction, we would like to see what happens at the small primes. For instance, an easy generalization of the theory above gives a bound

$$\text{spi}(\alpha) \mid (\text{per}(\alpha))^{c+3}$$

when $d < 2 \min_{q|n}(q) + 1$, by extending Corollary 5.2.

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