PRELIMINARY EXAM : APPLICATIONS ORIENTED MATH

1. Find the leading term in the asymptotic expansion of the following integral, in the indicated limits

$$I(\lambda) = \int_0^{\pi/2} \sqrt{t}\ \exp\left[\lambda(\sin t + \cos t)\right]\ dt.$$

(a) $\lambda \to +\infty$ (b) $\lambda \to -\infty$.

2. Consider the following integral as $\lambda \to +\infty$

$$\int_C \exp\left[\lambda\left(\frac{z^2}{2} - \frac{z^4}{4}\right)\right] dz.$$

(a) Find all saddle points in the complex z plane.

(b) At each saddle, find the local steepest descent (SD) and steepest ascent (SA) directions.

(c) Find the (global) steepest descent contour though each saddle.

(d) If C is the real axis, where z goes from $-\infty$ to $+\infty$, which saddle(s) determine the asymptotic behavior of the integral ? What if C is the imaginary axis, where z goes from $-i\infty$ to $+i\infty$?

3. Consider the ODE

$$y''(x) = \left(x^2 + \frac{1}{x^2}\right) y(x).$$

(a) Classify the points $x = 0$ and $x = \infty$ as ordinary, regular singular or irregular singular.

(b) Find the asymptotic behaviors of all solutions, in the limit $x \to 0$.

(c) Find the asymptotic behaviors of all solutions, in the limit $x \to \infty$.

4. Consider the boundary value problem

$$y''(x) + \lambda^2(1+x)y(x) = 0, \qquad 0 < x < 1$$

$$y(0) = 0, \qquad y'(1) - \lambda\sqrt{6}y(1) = 0.$$

Find the asymptotic behavior of the large eigenvalues and the corresponding eigenfunctions.

5. Consider the equation

$$x^2 \exp(-x^2) = \varepsilon.$$

(a) Assume that ε is positive and sufficiently small. How many solutions does this equation have in the range $x > 0$?

(b) Find two terms in the asymptotic expansions of all solutions, in the limit $\varepsilon \to 0^+$.

6. Consider the following oscillator with non-linear damping :

$$y''(t) + y(t) + \frac{\varepsilon}{3}[y'(t)]^3 = 0,$$

$$y(0) = 1, \qquad y'(0) = 0.$$

Use the two-time method to find an approximation to $y(t)$, valid up to times $t = O(\varepsilon^{-1})$.

7. Consider the following singularly perturbed boundary value problem :

$$\varepsilon y''(x) + x^{2/3} y'(x) + y(x) = 1,$$

$$y(0) = 0, \qquad y(1) = 2.$$

Construct an outer solution, locate any boundary layers and give their thickness, and then obtain a one term composite approximation.

8. Consider the following singularly perturbed boundary value problem, defined in the upper semicircle $\mathcal{D} = \{(x, y) : x^2 + y^2 < 1, y > 0\}$

$$\varepsilon \Delta u + u_x = \varepsilon [u_{xx} + u_{yy}] + u_x = 0, \qquad (x, y) \in \mathcal{D}$$

$$u(x, 0) = f(x), \qquad -1 < x < 1$$

$$u = g(\theta), \qquad r = \sqrt{x^2 + y^2} = 1, \quad 0 < \theta < \pi.$$

Construct an outer solution, locate any boundary layers and give their thickness. Give the ODE/PDE that applies in each layer, but you need not solve it.