Chapter Preview

We now make a significant departure from previous chapters by stepping out of the xy-plane into three-dimensional space. The fundamental concept of a *vector*—a quantity with magnitude and direction—is introduced in two and three dimensions. We then put vectors in motion by introducing *vector-valued functions*, or simply *vector functions*. The calculus of vector functions is a direct extension of everything you already know about limits, derivatives, and integrals. Also, with the calculus of vector functions, we can solve a wealth of practical problems involving the motion of objects in space. The chapter closes with an exploration of arc length, curvature, and tangent and normal vectors, all important features of space curves.

11.1 Vectors in the Plane

Imagine a raft drifting down a river, carried by the current. The speed and direction of the raft at a point may be represented by an arrow (Figure 11.1). The length of the arrow represents the speed of the raft at that point; longer arrows correspond to greater speeds. The orientation of the arrow gives the direction in which the raft is headed at that point. The arrows at points A and C in Figure 11.1 have the same length and direction indicating that the raft has the same speed and heading at these locations. The arrow at B is shorter and points to the left, indicating that the raft slows down as it nears the rock.

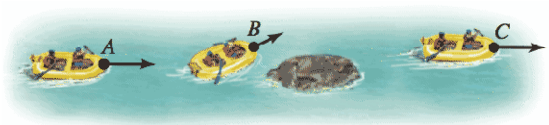


FIGURE 11.1

Basic Vector Operations

Scalar Multiplication

Vector Addition and Subtraction

Vector Components

Magnitude

Vector Operations in Terms of Components

Unit Vectors

Properties of Vector Operations

Applications of Vectors

Quick Quiz

SECTION 11.1 EXERCISES

Review Questions

1. Interpret the following statement: Points have a location but no size or direction; nonzero vectors have a size and direction, but no location.

2. What is a position vector?

3. Draw x- and y-axes on a page and mark two points P and Q. Then draw $\overrightarrow{PQ}$ and $\overrightarrow{QP}$.

4. On the diagram of Exercise 3, draw the position vector that is equal to $\overrightarrow{PQ}$.

5. Given a position vector $\mathbf{v}$, why are there infinitely many vectors equal to $\mathbf{v}$?

6. Explain how to add two vectors geometrically.

7. Explain how to find a scalar multiple of a vector geometrically.

8. Given two points P and Q, how are the components of $\overrightarrow{PQ}$ determined?

9. If $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$, how do you find $\mathbf{u} + \mathbf{v}$?

10. If $\mathbf{v} = \langle v_1, v_2 \rangle$ and c is a scalar, how do you find $c\,\mathbf{v}$?

11. How do you compute the magnitude of $\mathbf{v} = \langle v_1, v_2 \rangle$?

12. Express the vector $\mathbf{v} = \langle v_1, v_2 \rangle$ in terms of the unit vectors $\mathbf{i}$ and $\mathbf{j}$.

13. How do you compute $|\overrightarrow{PQ}|$ from the coordinates of the points P and Q?

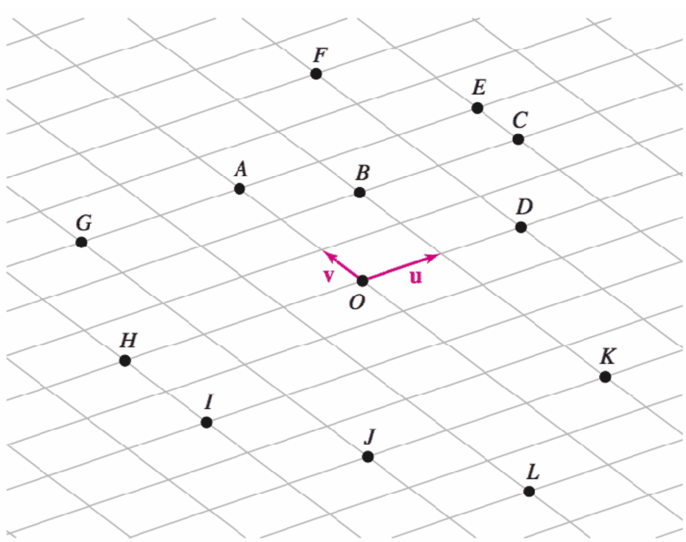
14. Explain how to find two unit vectors parallel to a vector $\mathbf{v}$.

15. How do you find a vector of length 10 in the direction of $\mathbf{v} = \langle 3, -2 \rangle$?

16. If a force has magnitude 100 and is directed 45 ° south of east, what are its components?

Basic Skills

17-20. Vector operations *Refer to the figure and carry out the following vector operations.*



17. Scalar multiples Write the following vectors as scalar multiples of $\mathbf{u}$ or $\mathbf{v}$.

a. $\overrightarrow{OA}$

b. $\overrightarrow{OD}$

c. $\overrightarrow{OH}$

d. $\overrightarrow{AG}$

e. $\overrightarrow{CE}$

18. Scalar multiples Write the following vectors as scalar multiples of **u** or **v**.

a. $\overrightarrow{IH}$

b. $\overrightarrow{HI}$

c. $\overrightarrow{JK}$

d. $\overrightarrow{FD}$

e. $\overrightarrow{EA}$

19. Vector addition Write the following vectors as sums of scalar multiples of **u** and **v**.

a. $\overrightarrow{OE}$

b. $\overrightarrow{OB}$

c. $\overrightarrow{OF}$

d. $\overrightarrow{OG}$

e. $\overrightarrow{OC}$

f. $\overrightarrow{OI}$

g. $\overrightarrow{OJ}$

h. $\overrightarrow{OK}$

i. $\overrightarrow{OL}$

20. Vector addition Write the following vectors as sums of scalar multiples of **u** and **v**.

a. $\overrightarrow{BF}$

b. $\overrightarrow{DE}$

c. $\overrightarrow{AF}$

d. $\overrightarrow{AD}$

e. $\overrightarrow{CD}$

f. $\overrightarrow{JD}$

g. $\overrightarrow{JI}$

h. $\overrightarrow{DB}$

i. $\overrightarrow{IL}$

21. Components and magnitudes Define the points $O(0, 0)$, $P(3, 2)$, $Q(4, 2)$, and $R(-6, -1)$. For each vector, do the following.

i. Sketch the vector in an xy-coordinate system.

ii. Compute the magnitude of the vector.

a. $\overrightarrow{OP}$

b. $\overrightarrow{QP}$

c. $\overrightarrow{RQ}$

22-25. Components and equality *Define the points* $P(-3, -1)$, $Q(-1, 2)$, $R(1, 2)$, $S(3, 5)$, $T(4, 2)$, *and* $U(6, 4)$.

22. Sketch $\overrightarrow{PU}$, $\overrightarrow{TR}$, and $\overrightarrow{SQ}$ and the corresponding position vectors.

23. Sketch $\overrightarrow{QU}$, $\overrightarrow{PT}$, and $\overrightarrow{RS}$ and the corresponding position vectors.

24. Find the equal vectors among $\overrightarrow{PQ}$, $\overrightarrow{RS}$, and $\overrightarrow{TU}$.

25. Which of the vectors $\overrightarrow{QT}$ or $\overrightarrow{SU}$ is equal to $\langle 5, 0\rangle$?

26-31. Vector operations *Let* $\mathbf{u} = \langle 4, -2\rangle$, $\mathbf{v} = \langle -4, 6\rangle$, *and* $\mathbf{w} = \langle 0, 8\rangle$. *Express the following vectors in the form* $\langle a, b\rangle$.

26. $\mathbf{u} + \mathbf{v}$

27. $\mathbf{w} - \mathbf{u}$

28. $2\,\mathbf{u} + 3\,\mathbf{v}$

29. $\mathbf{w} - 3\,\mathbf{v}$

30. $10\,\mathbf{u} - 3\,\mathbf{v} + \mathbf{w}$

31. $8\,\mathbf{w} + \mathbf{v} - 6\,\mathbf{u}$

32-37. Vector operations *Let* $\mathbf{u} = \langle 8, -4\rangle$, $\mathbf{v} = \langle 2, 6\rangle$, *and* $\mathbf{w} = \langle 5, 0\rangle$. *Carry out the following computations.*

32. Find $|\mathbf{u} + \mathbf{v} + \mathbf{w}|$.

33. Find $|2\,\mathbf{u} + 3\,\mathbf{v} - 4\,\mathbf{w}|$.

34. Find two vectors parallel to **u** with four times the magnitude of **u**.

35. Find two vectors parallel to **v** with three times the magnitude of **v**.

36. Which has the greatest magnitude, **u**, $3\,\mathbf{v}/2$, or $2\,\mathbf{w}$?

37. Which has the greater magnitude, $\mathbf{u} - \mathbf{v}$ or $\mathbf{w} - \mathbf{u}$?

38-43. Unit vectors *Define the points* $P(-4, 1)$, $Q(3, -4)$, *and* $R(2, 6)$. *Carry out the following calculations.*

38. Express $\overrightarrow{PQ}$ in the form $a\,\mathbf{i} + b\,\mathbf{j}$.

39. Express $\overrightarrow{QR}$ in the form $a\,\mathbf{i} + b\,\mathbf{j}$.

40. Find the unit vector with the same direction as $\overrightarrow{QR}$.

41. Find a unit vector parallel to $\overrightarrow{PR}$.

42. Find two vectors parallel to $\overrightarrow{RP}$ with length 4.

43. Find two vectors parallel to $\overrightarrow{QP}$ with length 4.

44. Parachute in the wind In still air, a parachute with a payload would fall vertically at a terminal speed of 40 m/s. Find the direction and magnitude of its terminal velocity relative to the ground if it falls in a steady wind blowing horizontally from west to east at 10 m/s.

45. Airplane in a wind An airplane flies horizontally from east to west at 320 mi/hr relative to the air. If it flies in a steady 40 mi/hr wind that blows horizontally toward the southwest (45° south of west), find the speed and direction of the airplane relative to the ground.

46. Canoe in a current A woman in a canoe paddles due west at 4 mi/hr relative to the water in a current that flows northwest at 2 mi/hr. Find the speed and direction of the canoe relative to the shore.

47. Boat in a wind A sailboat floats in a current that flows due east at 4 m/s. Due to a wind, the boat's actual speed relative to the shore is $4\sqrt{3}$ m/s in a direction 30° north of east. Find the speed and direction of the wind.

48. Towing a boat A boat is towed with a force of 150 lb with a rope that makes an angle of 30° to the horizontal. Find the horizontal and vertical components of the force.

49. Pulling a suitcase Suppose you pull a suitcase with a strap that makes a 60° angle with the horizontal. The magnitude of the force you exert on the suitcase is 40 lb.

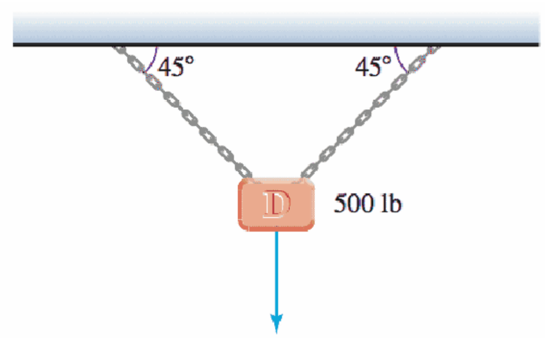
a. Find the horizontal and vertical components of the force.

b. Is the horizontal component of the force greater if the angle of the strap is 45° instead of 60°?

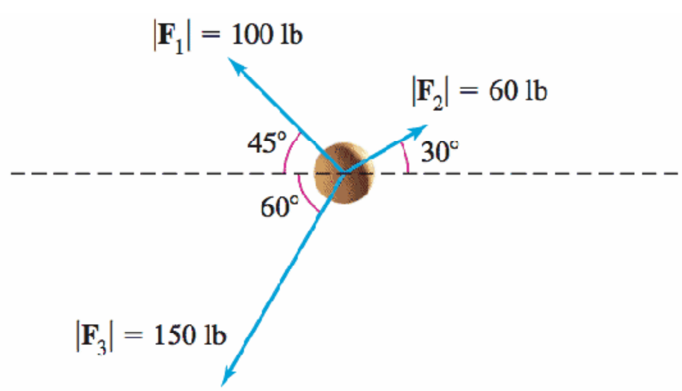
c. Is the vertical component of the force greater if the angle of the strap is 45° instead of 60°?

50. Which is greater? Which has a greater horizontal component, a 100-N force directed at an angle of 60° above the horizontal or a 60-N force directed at an angle of 30° above the horizontal?

51. Suspended load If a 500-lb load is suspended by two chains (see figure), what is the magnitude of the force each chain must be able to withstand?



52. Net force Three forces are applied to an object, as shown in the figure. Find the magnitude and direction of the sum of the forces.



Further Explorations

53. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. José travels from point A to point B in the plane by following vector $\mathbf{u}$, then vector $\mathbf{v}$, and then vector $\mathbf{w}$. If he starts at A and follows $\mathbf{w}$, then $\mathbf{v}$, and then $\mathbf{u}$, he still arrives at B.

b. Maria travels from A to B in the plane by following the vector $\mathbf{u}$. By following $-\mathbf{u}$, she returns from B to A.

c. The magnitude of $\mathbf{u} + \mathbf{v}$ is at least the magnitude of $\mathbf{u}$.

d. The magnitude of $\mathbf{u} + \mathbf{v}$ is at least the magnitude of $\mathbf{u}$ plus the magnitude of $\mathbf{v}$.

e. Parallel vectors have the same length.

f. If $\overrightarrow{AB} = \overrightarrow{CD}$, then $A = C$ and $B = D$.

g. If $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, then $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$.

h. If $\mathbf{u}$ and $\mathbf{v}$ are parallel and have the same direction, then $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$.

54. Finding vectors from two points Given the points $A(-2, 0)$, $B(6, 16)$, $C(1, 4)$, $D(5, 4)$, $E\left(\sqrt{2}, \sqrt{2}\right)$, and $F\left(3\sqrt{2}, -4\sqrt{2}\right)$, find the position vector equal to the following vectors.

a. $\overrightarrow{AB}$

b. $\overrightarrow{AC}$

c. $\overrightarrow{EF}$

d. $\overrightarrow{CD}$

55. Unit vectors

a. Find two unit vectors parallel to $\mathbf{v} = 6\,\mathbf{i} - 8\,\mathbf{j}$.

b. Find b if $\mathbf{v} = \langle 1/3, b \rangle$ is a unit vector.

c. Find all values of a such that $\mathbf{w} = a\,\mathbf{i} - \frac{a}{3}\,\mathbf{j}$ is a unit vector.

56. Equal vectors For the points $A(3, 4)$, $B(6, 10)$, $C(a + 2, b + 5)$, and $D(b + 4, a - 2)$, find the values of a and b such that $\overrightarrow{AB} = \overrightarrow{CD}$.

57-60. Vector equations *Use the properties of vectors to solve the following equations for the unknown vector* $\mathbf{x} = \langle a, b \rangle$. *Let* $\mathbf{u} = \langle 2, -3 \rangle$ *and* $\mathbf{v} = \langle -4, 1 \rangle$.

57. $10\,\mathbf{x} = \mathbf{u}$

58. $2\,\mathbf{x} + \mathbf{u} = \mathbf{v}$

59. $3\,\mathbf{x} - 4\,\mathbf{u} = \mathbf{v}$

60. $-4\,\mathbf{x} = \mathbf{u} - 8\,\mathbf{v}$

61-63. Linear Combinations *A sum of scalar multiples of two or more vectors (such as* $c_1\,\mathbf{u} + c_2\,\mathbf{v} + c_3\,\mathbf{w}$, *where* c_i *are scalars) is called a* ***linear combination*** *of the vectors. Let* $\mathbf{i} = \langle 1, 0 \rangle$, $\mathbf{j} = \langle 0, 1 \rangle$, $\mathbf{u} = \langle 1, 1 \rangle$, *and* $\mathbf{v} = \langle -1, 1 \rangle$.

61. Express $\langle 4, -8 \rangle$ as a linear combination of $\mathbf{i}$ and $\mathbf{j}$ (that is, find scalars c_1 and c_2 such that $\langle 4, -8 \rangle = c_1\,\mathbf{i} + c_2\,\mathbf{j}$).

62. Express $\langle 4, -8 \rangle$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

63. For arbitrary real numbers a and b, express $\langle a, b \rangle$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

64-65. Solving vector equations *Solve the following pairs of equations for the vectors* $\mathbf{u}$ *and* $\mathbf{v}$. *Assume* $\mathbf{i} = \langle 1, 0 \rangle$ *and* $\mathbf{j} = \langle 0, 1 \rangle$.

64. $2\mathbf{u} = \mathbf{i},\ \ \mathbf{u} - 4\mathbf{v} = \mathbf{j}$

65. $2\mathbf{u} + 3\mathbf{v} = \mathbf{i},\ \ \mathbf{u} - \mathbf{v} = \mathbf{j}$

66-69. Designer vectors *Find the following vectors.*

66. The vector that is 3 times $\langle 3, -5\rangle$ plus -9 times $\langle 6, 0\rangle$.

67. The vector in the direction of $\langle 5, -12\rangle$ with length 3.

68. The vector in the direction opposite to that of $\langle 6, -8\rangle$ with length 10.

69. The position vector for your final location if you start at the origin and walk along $\langle 4, -6\rangle$ followed by $\langle 5, 9\rangle$.

Applications

70. Ant on a page An ant is walking due east at a constant speed of 2 mi/hr on a sheet of paper that rests on a table. Suddenly the sheet of paper starts moving southeast at $\sqrt{2}$ mi/hr. Describe the motion of the ant relative to the table.

71. Clock vectors Consider the 12 vectors that have their tails at the center of a (circular) clock and their heads at the numbers on the edge of the clock.

a. What is the sum of these 12 vectors?

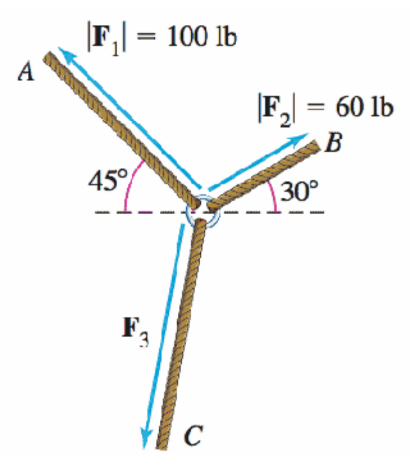
b. If the 12:00 vector is removed, what is the sum of the remaining 11 vectors?

c. By removing one or more of these 12 clock vectors, explain how to make the sum of the remaining vectors as large as possible in magnitude.

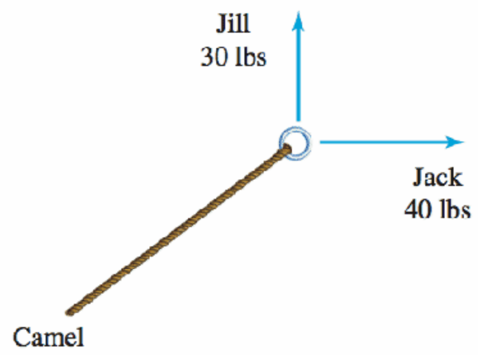
d. If the clock vectors originate at 12:00 and point to the other 11 numbers, what is the sum of the vectors?

(*Source: Calculus*, by Gilbert Strang. Wellesley-Cambridge Press, 1991.)

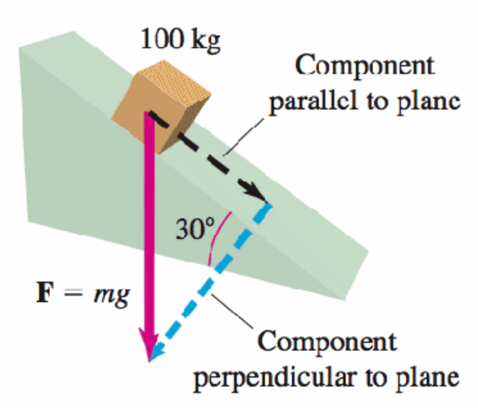
72. Three-way tug-of-war Three people located at A, B, and C pull on ropes tied to a ring. Find the magnitude and direction of the force with which C must pull so that no one moves (the system is at equilibrium).



73. Net force Jack pulls east on a rope attached to a camel with a force of 40 lb. Jill pulls north on a rope attached to the same camel with a force of 30 lb. What is the magnitude and direction of the force on the camel? Assume the vectors lie in a horizontal plane.



74. **Mass on a plane** A 100 –kg object rests on an inclined plane at an angle of 30 ° to the floor. Find the components of the force perpendicular to and parallel to the plane. (The vertical component of the force exerted by an object of mass m is its weight, which is $m\,g$, where $g = 9.8\ \mathrm{m/s^2}$ is the acceleration due to gravity.)



Additional Exercises

75-79. Vector properties *Prove the following vector properties using components. Then, make a sketch to illustrate the property geometrically. Suppose* **u**, **v**, *and* **w** *are vectors in the xy-plane and a and c are scalars.*

75. Commutative property: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

76. Associative property: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

77. Associative property: $a(c\,\mathbf{v}) = (a\,c)\,\mathbf{v}$

78. Distributive property 1: $a(\mathbf{u} + \mathbf{v}) = a\,\mathbf{u} + a\,\mathbf{v}$

79. Distributive property 2: $(a + c)\,\mathbf{v} = a\,\mathbf{v} + c\,\mathbf{v}$

80. **Midpoint of a line segment** Use vectors to show that the midpoint of the line segment joining $P(x_1,\ y_1)$ and $Q(x_2,\ y_2)$ is the point $((x_1 + x_2)/2,\ (y_1 + y_2)/2)$. (*Hint*: Let O be the origin and let M be the midpoint of PQ. Draw a picture and show that $\overrightarrow{OM} = \overrightarrow{OP} + \frac{1}{2}\overrightarrow{PQ} = \overrightarrow{OP} + \frac{1}{2}\left(\overrightarrow{OQ} - \overrightarrow{OP}\right)$.)

81. **Magnitude of scalar multiple** Prove that $|c\,\mathbf{v}| = |c|\,|\mathbf{v}|$, where c is a scalar and $\mathbf{v}$ is a vector.

82. **Equality of vectors** Assume $\overrightarrow{PQ}$ equals $\overrightarrow{RS}$. Does it follow that $\overrightarrow{PR}$ is equal to $\overrightarrow{QS}$? Prove your conclusion.

83. Linear independence A pair of nonzero vectors in the plane is *linearly dependent* if one vector is a scalar multiple of the other. Otherwise, the pair is *linearly independent*.

- **a.** Which pairs of the following vectors are linearly dependent and which are linearly independent: $\mathbf{u} = \langle 2, -3 \rangle$, $\mathbf{v} = \langle -12, 18 \rangle$, and $\mathbf{w} = \langle 4, 6 \rangle$?
- **b.** Explain geometrically what it means for a pair of vectors in the plane to be linearly dependent and independent.
- **c.** Prove that if a pair of vectors $\mathbf{u}$ and $\mathbf{v}$ is linearly independent, then given any vector $\mathbf{w}$, there are constants c_1 and c_2 such that $\mathbf{w} = c_1\,\mathbf{u} + c_2\,\mathbf{v}$.

84. Perpendicular vectors Show that two nonzero vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are perpendicular to each other if $u_1 v_1 + u_2 v_2 = 0$.

85. Parallel and perpendicular vectors Let $\mathbf{u} = \langle a, 5 \rangle$ and $\mathbf{v} = \langle 2, 6 \rangle$.

- **a.** Find the value of a such that $\mathbf{u}$ is parallel to $\mathbf{v}$.
- **b.** Find the value of a such that $\mathbf{u}$ is perpendicular to $\mathbf{v}$.

86. The Triangle Inequality Suppose $\mathbf{u}$ and $\mathbf{v}$ are vectors in the plane.

- **a.** Use the Triangle Rule for adding vectors to explain why $|\mathbf{u} + \mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$. This result is known as the *Triangle Inequality*.
- **b.** Under what conditions is $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$?