11.2 Vectors in Three Dimensions

Up to this point, our study of calculus has been limited to functions, curves, and vectors that can be plotted in the two-dimensional *xy*-plane. However, a two-dimensional coordinate system is insufficient for modeling many physical phenomena. For example, to describe the trajectory of a jet gaining altitude, we need two coordinates, say x and y, to measure east—west and north—south distances. In addition, another coordinate, say z, is needed to measure the altitude of the jet. By adding a third coordinate and creating an ordered triple (x, y, z), the location of the jet can be described. The set of all points described by the triples (x, y, z) is called *three-dimensional space*, *xyz*-space, or $\mathbb{R}^3$. Many of the properties of *xyz*-space are extensions of familiar ideas you have seen in the *xy*-plane.

Note

The *xyz*-Coordinate System

Equations of Simple Planes

Distances in *xyz*-Space

Equation of a Sphere

Vectors in $\mathbb{R}^3$

Magnitude and Unit Vectors

Quick Quiz

SECTION 11.2 EXERCISES

Review Questions

1. Explain how to plot the point $(3, -2, 1)$ in $\mathbb{R}^3$.

2. What is the y-coordinate of all points in the xz-plane?

3. Describe the plane $x = 4$.

4. What position vector is equal to the vector from $(3, 5, -2)$ to $(0, -6, 3)$?

5. Let $\mathbf{u} = \langle 3, 5, -7 \rangle$ and $\mathbf{v} = \langle 6, -5, 1 \rangle$. Evaluate $\mathbf{u} + \mathbf{v}$ and $3\,\mathbf{u} - \mathbf{v}$.

6. What is the magnitude of a vector joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$?

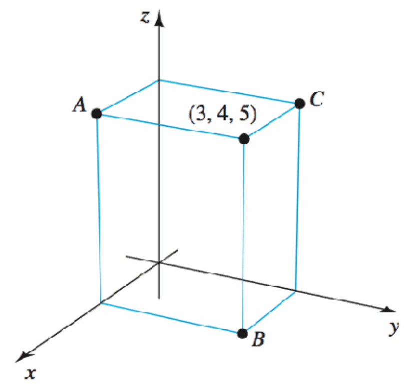
7. Which point is farther from the origin, $(3, -1, 2)$ or $(0, 0, -4)$?

8. Express the vector from $P(-1, -4, 6)$ to $Q(1, 3, -6)$ as a position vector in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Basic Skills

9-12. Points in $\mathbb{R}^3$ *Find the coordinates of the vertices A, B, and C of the following rectangular boxes.*

9.



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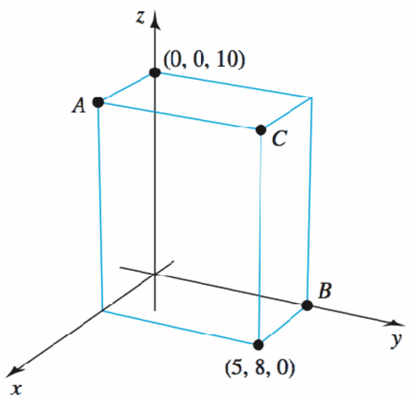
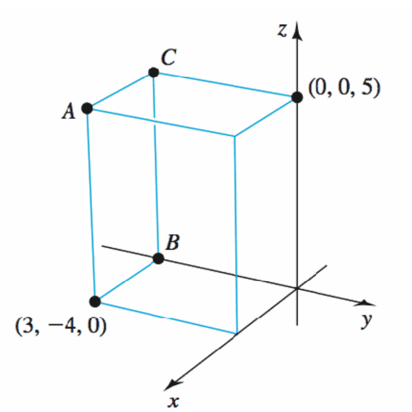
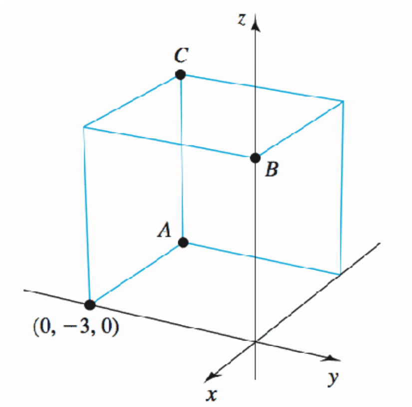


11.



12. Assume the all edges have the same length.



13-14. Plotting points in $\mathbb{R}^3$ *For each point $P(x, y, z)$ given below, let $A(x, y, 0)$, $B(x, 0, z)$, and $C(0, y, z)$ be points in the xy-, xz-, and yz-planes, respectively. Plot and label the points A, B, C, and P in $\mathbb{R}^3$.*

13. **a.** $P(2, 2, 4)$ **b.** $P(1, 2, 5)$ **c.** $P(-2, 0, 5)$

14. **a.** $P(-3, 2, 4)$ **b.** $P(4, -2, -3)$ **c.** $P(-2, -4, -3)$

15-20. Sketching planes *Sketch the following planes in the window* $[0, 5]\times[0, 5]\times[0, 5]$.

15. $x = 2$

16. $z = 3$

17. $y = 2$

18. $z = y$

19. The plane that passes through $(2, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 4)$

20. The plane parallel to the xz-plane containing the point $(1, 2, 3)$

21. **Planes** Sketch the plane parallel to the xy-plane through $(2, 4, 2)$ and find its equation.

22. **Planes** Sketch the plane parallel to the yz-plane through $(2, 4, 2)$ and find its equation.

23-26. Spheres and balls *Find an equation or inequality that describes the following objects.*

23. A sphere with center $(1, 2, 3)$ and radius 4

24. A sphere with center $(1, 2, 0)$ passing through the point $(3, 4, 5)$

25. A ball with center $(-2, 0, 4)$ and radius 1

26. A ball with center $(0, -2, 6)$ with the point $(1, 4, 8)$ on its boundary

27. **Midpoints and spheres** Find an equation of the sphere passing through $P(1, 0, 5)$ and $Q(2, 3, 9)$ with its center at the midpoint of PQ.

28. **Midpoints and spheres** Find an equation of the sphere passing through $P(-4, 2, 3)$ and $Q(0, 2, 7)$ with its center at the midpoint of PQ.

29-34. Identifying sets *Give a geometric description of the following sets of points.*

29. $x^2 + y^2 + z^2 - 2y - 4z - 4 = 0$

30. $x^2 + y^2 + z^2 - 6x + 6y - 8z - 2 = 0$

31. $x^2 + y^2 - 14y + z^2 \geq -13$

32. $x^2 + y^2 - 14y + z^2 \leq -13$

33. $x^2 + y^2 + z^2 - 8x - 14y - 18z \leq 65$

34. $x^2 + y^2 + z^2 - 8x + 14y - 18z \geq 65$

35-38. Vector operations *For the given vectors* **u** *and* **v**, *evaluate the following expressions.*

a. $3\mathbf{u} + 2\mathbf{v}$ **b.** $4\mathbf{u} - \mathbf{v}$ **c.** $|\mathbf{u} + 3\mathbf{v}|$

35. $\mathbf{u} = \langle 1, 3, 0 \rangle$, $\mathbf{v} = \langle 3, 0, 2 \rangle$

36. $\mathbf{u} = \langle -1, 1, 0 \rangle$, $\mathbf{v} = \langle 2, -4, 1 \rangle$

37. $\mathbf{u} = \langle -7, 5, 1 \rangle$, $\mathbf{v} = \langle -2, 4, 0 \rangle$

38. $\mathbf{u} = \left\langle 5, 1, 3\sqrt{2} \right\rangle$, $\mathbf{v} = \left\langle 2, 0, 7\sqrt{2} \right\rangle$

39-44. Unit vectors and magnitude *Consider the following points P and Q.*

a. *Find* $\overrightarrow{PQ}$ *and state your answer in two forms:* $\langle a, b, c \rangle$ *and* $a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$.

b. *Find the magnitude of* $\overrightarrow{PQ}$.

c. *Find two unit vectors parallel to* $\overrightarrow{PQ}$.

39. $P(1, 5, 0)$, $Q(3, 11, 2)$

40. $P(5, 11, 12)$, $Q(1, 14, 13)$

41. $P(-3, 1, 0)$, $Q(-3, -4, 1)$

42. $P(3, 8, 12)$, $Q(3, 9, 11)$

43. $P(0, 0, 2)$, $Q(-2, 4, 0)$

44. $P(a, b, c)$, $Q(1, 1, -1)$ (a, b, c are real numbers).

45. **Crosswinds** A small plane is flying horizontally due east in calm air at 250 mi/hr when it is hit by a horizontal crosswind blowing southwest at 50 mi/hr and a 30 mi/hr updraft. Find the resulting speed of the plane and describe with a sketch the approximate direction of the velocity relative to the ground.

46. **Combined force** An object at the origin is acted on by the forces $\mathbf{F}_1 = 20\,\mathbf{i} - 10\,\mathbf{j}$, $\mathbf{F}_2 = 30\,\mathbf{j} + 10\,\mathbf{k}$, and $\mathbf{F}_3 = 40\,\mathbf{j} + 20\,\mathbf{k}$. Find the magnitude of the combined force and describe the approximate direction of the force.

47. **Submarine course** A submarine climbs at an angle of 30° above the horizontal with a heading to the northeast. If its speed is 20 knots, find the components of the velocity in the east, north, and vertical directions.

48. **Maintaining equilibrium** An object is acted upon by the forces $\mathbf{F}_1 = \langle 10, 6, 3 \rangle$ and $\mathbf{F}_2 = \langle 0, 4, 9 \rangle$. Find the force $\mathbf{F}_3$ that must act on the object so that the sum of the forces is zero.

Further Explorations

49. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

- **a.** Suppose **u** and **v** both make a $45°$ angle with **w** in $\mathbb{R}^3$. Then, **u** + **v** makes a $45°$ angle with **w**.
- **b.** Suppose **u** and **v** both make a $90°$ angle with **w** in $\mathbb{R}^3$. Then, **u** + **v** can never make a $90°$ angle with **w**.
- **c.** $\mathbf{i} + \mathbf{j} + \mathbf{k} = \mathbf{0}$
- **d.** The intersection of the planes $x = 1$, $y = 1$, and $z = 1$ is a point.

50-52. Sets of points *Describe with a sketch the sets of points* (x, y, z) *satisfying the following equations.*

50. $(x + 1)(y - 3) = 0$

51. $x^2 y^2 z^2 > 0$

52. $y - z = 0$

53-56. Parallel vectors of varying lengths *Find vectors parallel to* **v** *of the given length.*

53. $\mathbf{v} = \langle 6, -8, 0 \rangle$; length = 20

54. $\mathbf{v} = \langle 3, -2, 6 \rangle$; length = 10

55. $\mathbf{v} = \overrightarrow{PQ}$ with $P(3, 4, 0)$ and $Q(2, 3, 1)$; length = 3

56. $\mathbf{v} = \overrightarrow{PQ}$ with $P(1, 0, 1)$ and $Q(2, -1, 1)$; length = 3

57. Collinear points Determine whether the points P, Q, and R are collinear (lie on a line) by comparing $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. If the points are collinear, determine which point lies between the other two points.

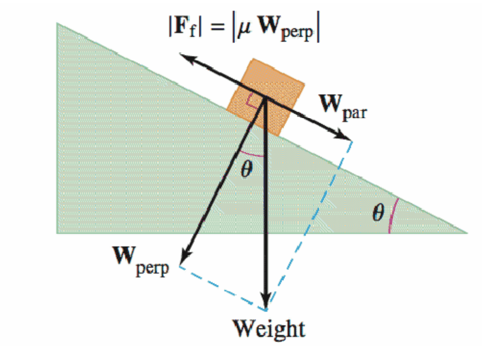
- **a.** $P(1, 6, -5)$, $Q(2, 5, -3)$, $R(4, 3, 1)$
- **b.** $P(1, 5, 7)$, $Q(5, 13, -1)$, $R(0, 3, 9)$
- **c.** $P(1, 2, 3)$, $Q(2, -3, 6)$, $R(3, -1, 9)$
- **d.** $P(9, 5, 1)$, $Q(11, 18, 4)$, $R(6, 3, 0)$

58. Collinear points Determine the values of x and y such that the points $(1, 2, 3)$, $(4, 7, 1)$, and $(x, y, 2)$ are collinear (lie on a line).

59. Lengths of the diagonals of a box A fisherman wants to know if his fly rod will fit in a rectangular $2\text{ ft} \times 3\text{ ft} \times 4\text{ ft}$ packing box. What is the longest rod that fits in this box?

Applications

T **60. Forces on an inclined plane** An object on an inclined plane does not slide provided the component of the object's weight parallel to the plane $|\mathbf{W}_{\text{par}}|$ is less than or equal to the magnitude of the opposing frictional force $|\mathbf{F}_{\text{f}}|$. The magnitude of the frictional force, in turn, is proportional to the component of the object's weight perpendicular to the plane $|\mathbf{W}_{\text{perp}}|$ (see figure). The constant of proportionality is the coefficient of static friction, μ.

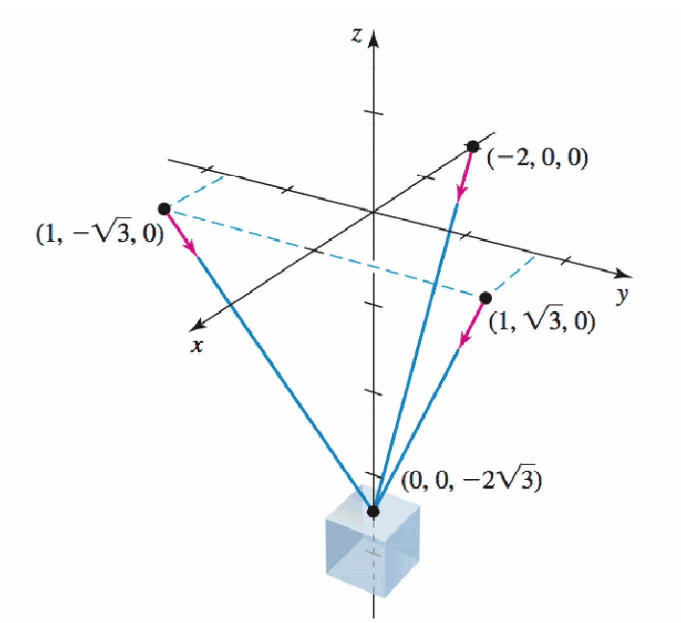


a. Suppose a 100-lb block rests in a plane that is tilted at an angle of $\theta = 20°$ to the horizontal. Find $|\mathbf{W}_{par}|$ and $|\mathbf{W}_{perp}|$.

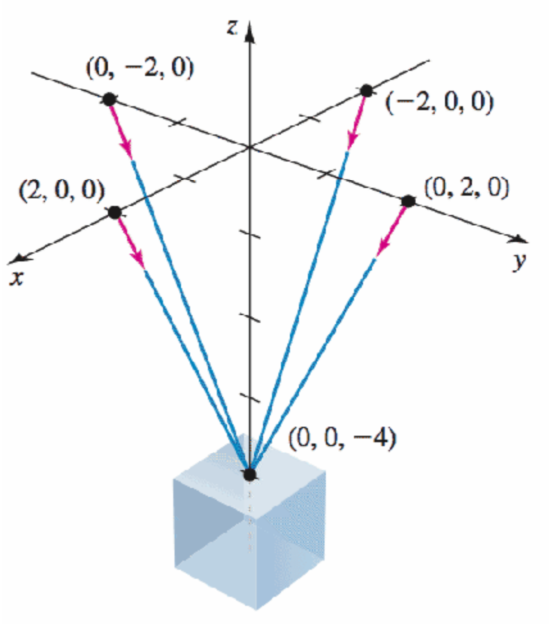
b. The condition for the block not sliding is $|\mathbf{W}_{par}| \le \mu |\mathbf{W}_{perp}|$. If $\mu = 0.65$, does the block slide?

c. What is the critical angle above which the block slides?

61. Three-cable load A 500-lb load hangs from three cables of equal length that are anchored at the points $(-2, 0, 0)$, $\left(1, \sqrt{3}, 0\right)$, and $\left(1, -\sqrt{3}, 0\right)$. The load is located at $\left(0, 0, -2\sqrt{3}\right)$. Find the vectors describing the forces on the cables due to the load.



62. Four-cable load A 500-lb load hangs from four cables of equal length that are anchored at the points $(\pm 2, 0, 0)$ and $(0, \pm 2, 0)$. The load is located at $(0, 0, -4)$. Find the vectors describing the forces on the cables due to the load.



Additional Exercises

63. Possible parallelograms The points $O(0, 0, 0)$, $P(1, 4, 6)$, and $Q(2, 4, 3)$ lie at three vertices of a parallelogram. Find all possible locations of the fourth vertex.

64. Diagonals of parallelograms Two sides of a parallelogram are formed by the vectors **u** and **v**. Prove that the diagonals of the parallelogram are **u** + **v** and **u** − **v**.

65. Midpoint formula Prove that the midpoint of the line segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

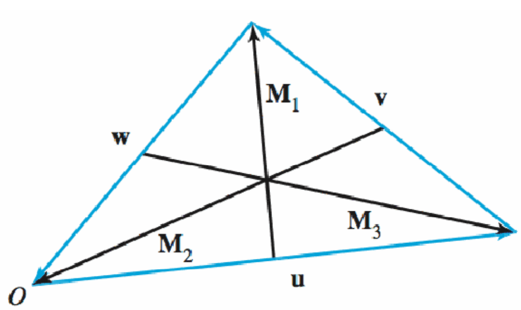
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

66. Equation of a sphere For constant a, b, c, and d, show that the equation

$$x^2 + y^2 + z^2 - 2ax - 2by - 2cz = d$$

describes a sphere centered at (a, b, c) with radius r, where $r^2 = d + a^2 + b^2 + c^2$, provided $d + a^2 + b^2 + c^2 > 0$.

67. Medians of a triangle—coordinate free Assume that **u**, **v**, and **w** are vectors in $\mathbb{R}^3$ that form the sides of a triangle (see figure). Use the following steps to prove that the medians intersect at a point that divides each median in a 2:1 ratio. The proof does not use a coordinate system.



a. Show that $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$.

b. Let $\mathbf{M}_1$ be the median vector from the midpoint of $\mathbf{u}$ to the opposite vertex. Define $\mathbf{M}_2$ and $\mathbf{M}_3$ similarly. Using the geometry of vector addition show that $\mathbf{M}_1 = \mathbf{u}/2 + \mathbf{v}$. Find analogous expressions for $\mathbf{M}_2$ and $\mathbf{M}_3$.

c. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be the vectors from O to the points one-third of the way along $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M}_3$, respectively. Show that $\mathbf{a} = \mathbf{b} = \mathbf{c} = (\mathbf{u} - \mathbf{w})/3$.

d. Conclude that the medians intersect at a point that divides each median in a 2:1 ratio.

68. Medians of a triangle—with coordinates In contrast to the proof in Exercise 67, we now use coordinates and position vectors to prove the same result. Without loss of generality, let $P(x_1, y_1, 0)$ and $Q(x_2, y_2, 0)$ be two points in the xy-plane and let $R(x_3, y_3, z_3)$ be a third point, such that P, Q, and R do not lie on a line. Consider $\triangle PQR$.

a. Let M_1 be the midpoint of the side PQ. Find the coordinates of M_1 and the components of the vector $\overrightarrow{RM_1}$.

b. Find the vector $\overrightarrow{OZ_1}$ from the origin to the point Z_1 two-thirds of the way along $\overrightarrow{RM_1}$.

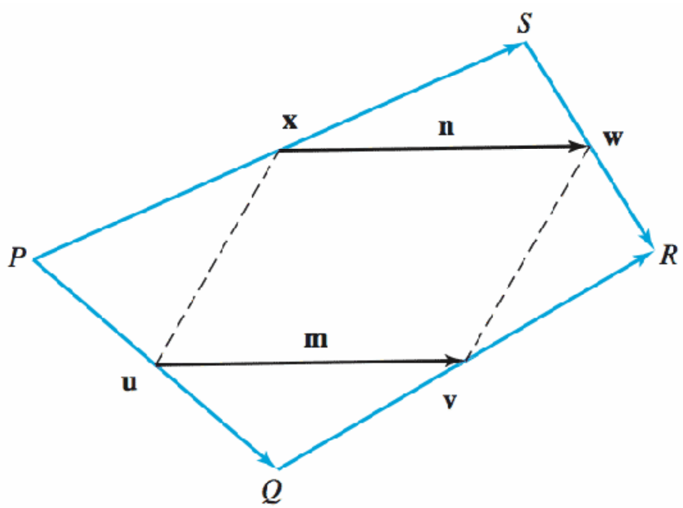
c. Repeat the calculation of part (b) with the midpoint M_2 of RQ and the vector $\overrightarrow{PM_2}$ to obtain the vector $\overrightarrow{OZ_2}$.

d. Repeat the calculation of part (b) with the midpoint M_3 of PR and the vector $\overrightarrow{QM_3}$ to obtain the vector $\overrightarrow{OZ_3}$.

e. Conclude that the medians of $\triangle PQR$ intersect at a point. Give the coordinates of the point.

f. With $P(2, 4, 0)$, $Q(4, 1, 0)$, $R(6, 3, 4)$, find the point at which the medians of $\triangle PQR$ intersect.

69. The amazing quadrilateral property—coordinate free The points P, Q, R, and S, joined by the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$ are the vertices of a quadrilateral in $\mathbb{R}^3$. *The four points needn't lie in a plane* (see figure). Use the following steps to prove that the line segments joining the midpoints of the sides of the quadrilateral form a parallelogram. The proof does not use a coordinate system.



a. Use vector addition to show that $\mathbf{u} + \mathbf{v} = \mathbf{w} + \mathbf{x}$.

b. Let $\mathbf{m}$ be the vector that joins the midpoints of PQ and QR. Show that $\mathbf{m} = (\mathbf{u} + \mathbf{v})/2$.

c. Let $\mathbf{n}$ be the vector that joins the midpoints of PS and SR. Show that $\mathbf{n} = (\mathbf{x} + \mathbf{w})/2$.

d. Combine parts (a), (b), and (c) to conclude that $\mathbf{m} = \mathbf{n}$.

e. Explain why part (d) implies that the line segments joining the midpoints of the sides of the quadrilateral form a parallelogram.

70. The amazing quadrilateral property—with coordinates Prove the quadrilateral property in Exercise 69 assuming the coordinates of P, Q, R, and S are $P(x_1, y_1, 0)$, $Q(x_2, y_2, 0)$, $R(x_3, y_3, 0)$, and $S(x_4, y_4, z_4)$, where we assume that P, Q, and R lie in the xy-plane without loss of generality.