11.3 Dot Products

The *dot product* is used to determine the angle between two vectors. It is also a tool for calculating *projections*—the measure of how much of a given vector lies in the direction of another vector.

Note

To see the usefulness of the dot product, consider an example. Recall that the work done by a constant force F in moving an object a distance d is $W = F\,d$ (Section 6.6). This rule applies provided the force acts in the direction of motion (Figure 11.43a). Now assume the force is a vector $\mathbf{F}$ applied at an angle θ to the direction of motion; the resulting displacement of the object is a vector $\mathbf{d}$. In this case, the work done by the force is the component of the force in the direction of motion multiplied by the distance moved by the object, which is $W = (|\mathbf{F}|\cos\theta)\,|\mathbf{d}|$ (Figure 11.43b). We call this product of the magnitudes of two vectors and the cosine of the angle between them the *dot product*.

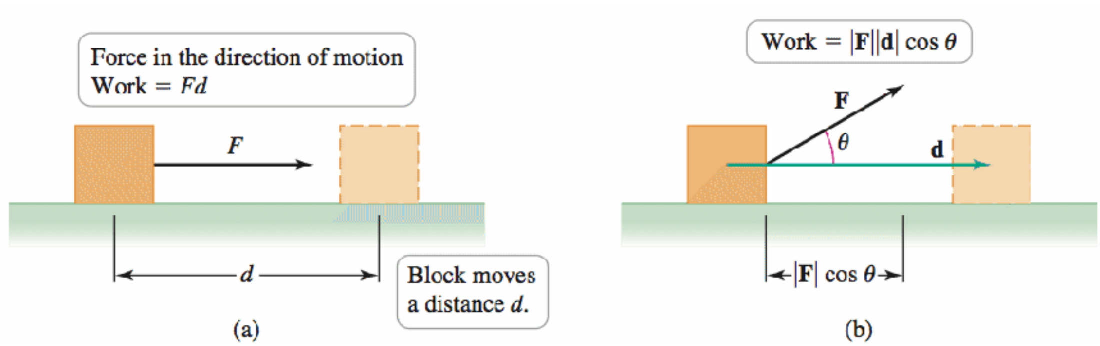


FIGURE 11.43

Note

Two Forms of the Dot Product

Orthogonal Projections

Applications of Dot Products

Quick Quiz

SECTION 11.3 EXERCISES

Review Questions

1. Define the dot product of **u** and **v** in terms of their magnitudes and the angle between them.

2. Define the dot product of **u** and **v** in terms of the components of the vectors.

3. Compute $\langle 2, 3, -6\rangle \cdot \langle 1, -8, 3\rangle$.

4. What is the dot product of two orthogonal vectors?

5. Explain how to find the angle between two nonzero vectors.

6. Use a sketch to illustrate the projection of **u** onto **v**.

7. Use a sketch to illustrate the scalar component of **u** in the direction of **v**.

8. Explain how the work done by a force in moving an object is computed using dot products.

Basic Skills

9-12. Dot product from the definition *Consider the following vectors* **u** *and* **v**. *Sketch the vectors, find the angle between the vectors, and compute the dot product using the definition* $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$.

9. $\mathbf{u} = 4\,\mathbf{i}$ and $\mathbf{v} = 6\,\mathbf{j}$

10. $\mathbf{u} = \langle -3, 2, 0\rangle$ and $\mathbf{v} = \langle 0, 0, 6\rangle$

11. $\mathbf{u} = \langle 10, 0\rangle$ and $\mathbf{v} = \langle 10, 10\rangle$

12. $\mathbf{u} = \left\langle -\sqrt{3}\,,\, 1\right\rangle$ and $\mathbf{v} = \left\langle \sqrt{3}\,,\, 1\right\rangle$

T **13-18. Dot products and angles** *Compute the dot product of the vectors* **u** *and* **v**, *and find the approximate angle between the vectors.*

13. $\mathbf{u} = 4\,\mathbf{i} + 3\,\mathbf{j}$ and $\mathbf{v} = 4\,\mathbf{i} - 6\,\mathbf{j}$

14. $\mathbf{u} = \langle 3, 4, 0\rangle$ and $\mathbf{v} = \langle 0, 4, 5\rangle$

15. $\mathbf{u} = \langle -10, 0, 4\rangle$ and $\mathbf{v} = \langle 1, 2, 3\rangle$

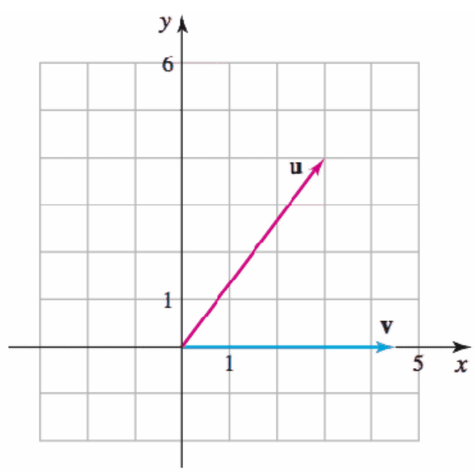
16. $\mathbf{u} = \langle 3, -5, 2\rangle$ and $\mathbf{v} = \langle -9, 5, 1\rangle$

17. $\mathbf{u} = 2\,\mathbf{i} - 3\,\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 4\,\mathbf{j} + 2\,\mathbf{k}$

18. $\mathbf{u} = \mathbf{i} - 4\,j - 6\,\mathbf{k}$ and $\mathbf{v} = 2\,\mathbf{i} - 4\,\mathbf{j} + 2\,\mathbf{k}$

19-22. Sketching orthogonal projections *Find* $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ *and* $\text{scal}_{\mathbf{v}}\,\mathbf{u}$ *by inspection without using formulas.*

19.



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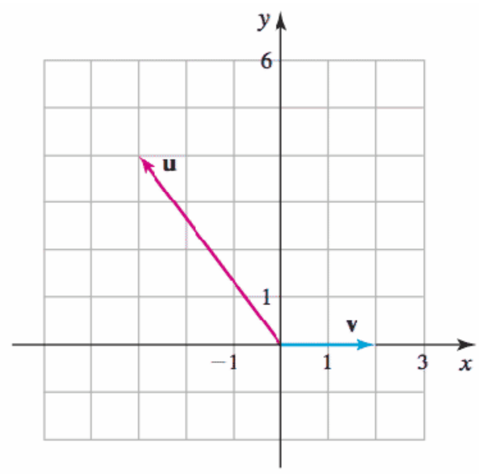

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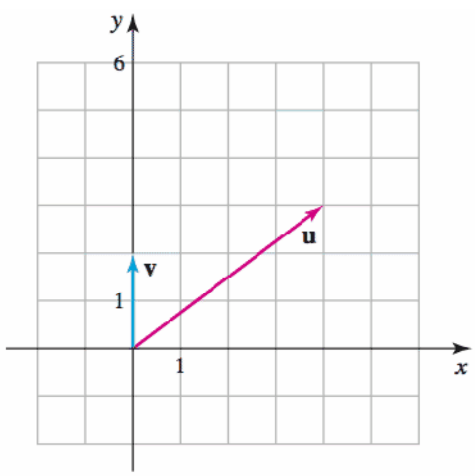

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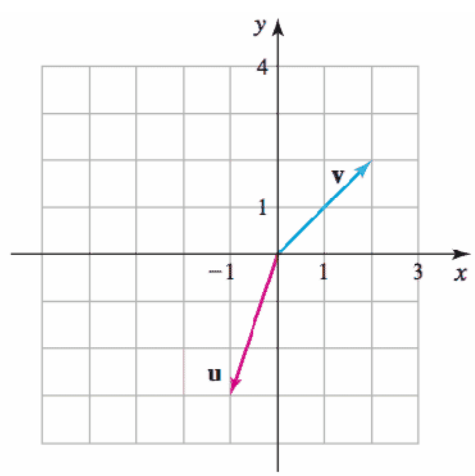


23-28. Calculating orthogonal projections *For the given vectors* **u** *and* **v**, *calculate* $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ *and* $\text{scal}_{\mathbf{v}}\,\mathbf{u}$,

23. $\mathbf{u} = \langle -1, 4\rangle$ and $\mathbf{v} = \langle -4, 2\rangle$

24. $\mathbf{u} = \langle 10, 5 \rangle$ and $\mathbf{v} = \langle 2, 6 \rangle$

25. $\mathbf{u} = \langle -8, 0, 2 \rangle$ and $\mathbf{v} = \langle 1, 3, -3 \rangle$

26. $\mathbf{u} = \langle 3, -5, 2 \rangle$ and $\mathbf{v} = \langle -9, 5, 1 \rangle$

27. $\mathbf{u} = 2\,\mathbf{i} - 4\,\mathbf{k}$ and $\mathbf{v} = 9\,\mathbf{i} + 2\,\mathbf{k}$

28. $\mathbf{u} = \mathbf{i} + 4\,\mathbf{j} + 7\,\mathbf{k}$ and $\mathbf{v} = 2\,\mathbf{i} - 4\,\mathbf{j} + 2\,\mathbf{k}$

29-32. Computing work *Calculate the work done in the following situations.*

29. A suitcase is pulled 50 ft along a flat sidewalk with a constant upward force of 30 lb at an angle of 30° with the horizontal.

T **30.** A stroller is pushed 20 m with a constant downward force of 10 N at an angle of 15° with the horizontal.

31. A constant force $\mathbf{F} = \langle 40, 30 \rangle$ (N) is used to move a sled horizontally 10 m.

32. A constant force $\mathbf{F} = \langle 2, 4, 1 \rangle$ moves an object from (0, 0, 0) to (2, 4, 6).

33-36. Parallel and normal forces *Find the components of the vertical force* $\mathbf{F} = \langle 0, -10 \rangle$ *in the directions parallel to and normal to the following planes. Show that the total force is the sum of the two component forces.*

33. A plane that makes an angle of $\pi/4$ with the positive x-axis.

34. A plane that makes an angle of $\pi/6$ with the positive x-axis.

35. A plane that makes an angle of $\pi/3$ with the positive x-axis.

36. A plane that makes an angle of $\theta = \tan^{-1}\left(\frac{4}{5}\right)$ with the positive x-axis.

Further Explorations

37. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

- **a.** $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \text{proj}_{\mathbf{u}}\,\mathbf{v}$.
- **b.** If nonzero vectors **u** and **v** have the same magnitude they make equal angles with $\mathbf{u} + \mathbf{v}$.
- **c.** $(\mathbf{u} \cdot \mathbf{i})^2 + (\mathbf{u} \cdot \mathbf{j})^2 + (\mathbf{u} \cdot \mathbf{k})^2 = |\mathbf{u}|^2$
- **d.** If **u** is orthogonal to **v** and **v** is orthogonal to **w**, then **u** is orthogonal to **w**.
- **e.** The vectors orthogonal to $\langle 1, 1, 1 \rangle$ lie on the same line.
- **f.** If $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \mathbf{0}$, then vectors **u** and **v** (both nonzero) are orthogonal.

38-42. Orthogonal vectors *Let a and b be real numbers.*

38. Find all unit vectors orthogonal to $\mathbf{v} = \langle 3, 4, 0 \rangle$.

39. Find all vectors $\langle 1, a, b \rangle$ orthogonal to $\langle 4, -8, 2 \rangle$.

40. Describe all unit vectors orthogonal to $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

41. Find three mutually orthogonal unit vectors in $\mathbb{R}^3$ besides $\pm\mathbf{i}$, $\pm\mathbf{j}$, and $\pm\mathbf{k}$.

42. Find two vectors that are orthogonal to $\langle 0, 1, 1 \rangle$ and to each other.

43. Equal angles Consider all unit position vectors **u** in $\mathbb{R}^3$ that make a $60°$ angle with the unit vector **k** in $\mathbb{R}^3$.

a. Prove that $\text{proj}_{\mathbf{k}}\,\mathbf{u}$ is the same for all vectors in this set.

b. Is $\text{scal}_{\mathbf{k}}\,\mathbf{u}$ the same for all vectors in this set?

44-47. Vectors with equal projections *Given a fixed vector* **v**, *there is an infinite set of vectors* **u** *with the same value of* $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.

44. Find another vector that has the same projection onto $\mathbf{v} = \langle 1, 1\rangle$ as $\mathbf{u} = \langle 1, 2\rangle$. Draw a picture.

45. Let $\mathbf{v} = \langle 1, 1\rangle$. Give a description of the position vectors **u** such that $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \text{proj}_{\mathbf{v}}\,\langle 1, 2\rangle$.

46. Find another vector that has the same projection onto $\mathbf{v} = \langle 1, 1, 1\rangle$ as $\mathbf{u} = \langle 1, 2, 3\rangle$.

47. Let $\mathbf{v} = \langle 0, 0, 1\rangle$. Give a description of all position vectors **u** such that $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \text{proj}_{\mathbf{v}}\,\langle 1, 2, 3\rangle$.

48-51. Decomposing vectors *For the following vectors* **u** *and* **v**, *express* **u** *as the sum* $\mathbf{u} = \mathbf{p} + \mathbf{n}$, *where* **p** *is parallel to* **v** *and* **n** *is orthogonal to* **v**.

48. $\mathbf{u} = \langle 4, 3\rangle$, $\mathbf{v} = \langle 1, 1\rangle$

49. $\mathbf{u} = \langle -2, 2\rangle$, $\mathbf{v} = \langle 2, 1\rangle$

50. $\mathbf{u} = \langle 4, 3, 0\rangle$, $\mathbf{v} = \langle 1, 1, 1\rangle$

51. $\mathbf{u} = \langle -1, 2, 3\rangle$, $\mathbf{v} = \langle 2, 1, 1\rangle$

52-55. Distance between a point and a line *Carry out the following steps to determine the distance between the point P and the line l through the origin.*

a. *Find any vector* **v** *in the direction of l.*

b. *Find the position vector* **u** *corresponding to P.*

c. *Find* $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.

d. *Show that* $\mathbf{w} = \mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u}$ *is a vector orthogonal to* **v** *whose length is the distance between P and the line l.*

e. *Find* **w** *and* $|\mathbf{w}|$. *Explain why* $|\mathbf{w}|$ *is the distance between P and l.*

52. $P(2, -5)$; $l : y = 3x$

53. $P(-12, 4)$; $l : y = 2x$

54. $P(0, 2, 6)$; l has the direction of $\langle 3, 0, -4\rangle$.

55. $P(1, 1, -1)$; l has the direction of $\langle -6, 8, 3\rangle$.

56-58. Orthogonal unit vectors in the xy-plane *Consider the vectors* $\mathbf{I} = \left\langle 1/\sqrt{2}, 1/\sqrt{2}\right\rangle$ *and* $\mathbf{J} = \left\langle -1/\sqrt{2}, 1/\sqrt{2}\right\rangle$.

56. Show that **I** and **J** are orthogonal unit vectors.

57. Express **I** and **J** in terms of the usual unit coordinate vectors **i** and **j**. Then write **i** and **j** in terms of **I** and **J**.

58. Write the vector $\langle 2, -6\rangle$ in terms of **I** and **J**.

59. Orthogonal unit vectors in $\mathbb{R}^3$ Consider the vectors $\mathbf{I} = \left\langle 1/2,\ 1/2,\ 1\Big/\sqrt{2} \right\rangle$, $\mathbf{J} = \left\langle -1\Big/\sqrt{2},\ 1\Big/\sqrt{2},\ 0 \right\rangle$, and $\mathbf{K} = \left\langle 1/2,\ 1/2,\ -1\Big/\sqrt{2} \right\rangle$.

a. Sketch **I**, **J**, and **K** and show that they are unit vectors.

b. Show that **I**, **J**, and **K** are mutually orthogonal.

a. Express the vector $\langle 1,\ 0,\ 0\rangle$ in terms of **I**, **J**, and **K**.

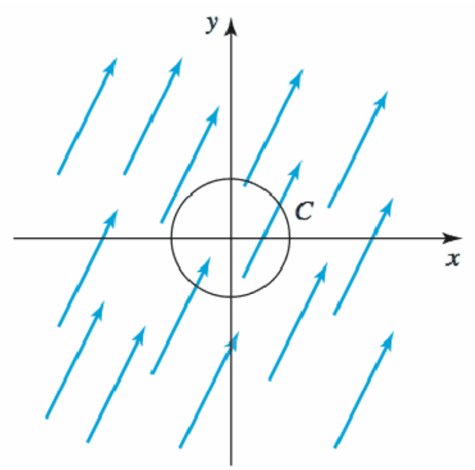
T **60-61. Angles of a triangle** *For the given points P, Q, and R, find the approximate measurements of the angles of* $\triangle PQR$.

60. $P(1,\ -4)$, $Q(2,\ 7)$, $R(-2,\ 2)$

61. $P(0,\ -1,\ 3)$, $Q(2,\ 2,\ 1)$, $R(-2,\ 2,\ 4)$

Applications

62. Flow through a circle Suppose water flows in a thin sheet over the xy-plane with a uniform velocity given by the vector $\mathbf{v} = \langle 1,\ 2\rangle$; this means that at all points of the plane, the velocity of the water has components 1 m/s in the x-direction and 2 m/s in the y-direction (see figure). Let C be an imaginary unit circle (that does not interfere with the flow).



a. Show that at the point $(x,\ y)$ on the circle C the outward pointing unit vector normal to C is $\mathbf{n} = \langle x,\ y\rangle$.

b. Show that at the point $(\cos\theta,\ \sin\theta)$ on the circle C the outward-pointing unit vector normal to C is also $\mathbf{n} = \langle \cos\theta,\ \sin\theta\rangle$.

c. Find all points on C at which the velocity is normal to C.

d. Find all points on C at which the velocity is tangential to C.

e. At each point on C find the component of **v** normal to C. Express the answer as a function of $(x,\ y)$ and as a function of θ.

f. What is the net flow through the circle? That is, does water accumulate inside the circle?

63. Heat flux Let D be a solid heat-conducting cube formed by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$. The heat flow at every point of D is given by the constant vector $\mathbf{Q} = \langle 0,\ 2,\ 1\rangle$.

a. Through which faces of D does **Q** point into D?

b. Through which faces of D does **Q** point out of D?

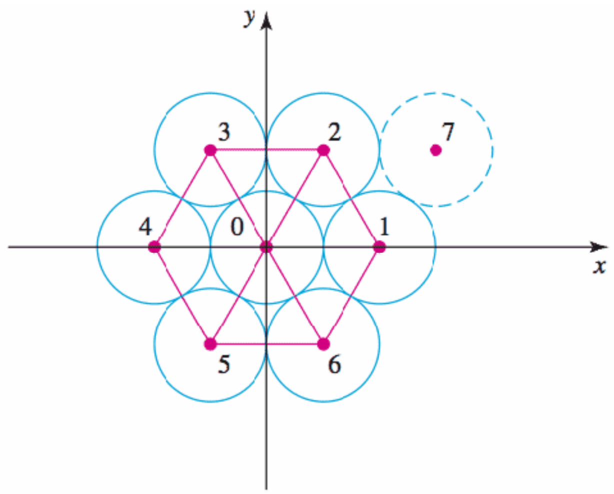
c. On which faces of D is **Q** tangential to D (pointing neither in nor out of D)?

d. Find the scalar component of **Q** normal to the face $x = 0$.

e. Find the scalar component of **Q** normal to the face $z = 1$.

f. Find the scalar component of **Q** normal to the face $y = 0$.

64. Hexagonal circle packing The German mathematician Gauss proved that the densest way to pack circles with the same radius in the plane is to place the centers of the circles on a hexagonal grid (see figure). Some molecular structures use this packing or its three-dimensional analog. Assume all circles have a radius of 1 and let $\mathbf{r}_{ij}$ be the vector that extends from the center of circle i to the center of circle j for $i, j = 0, 1, \ldots, 6$.

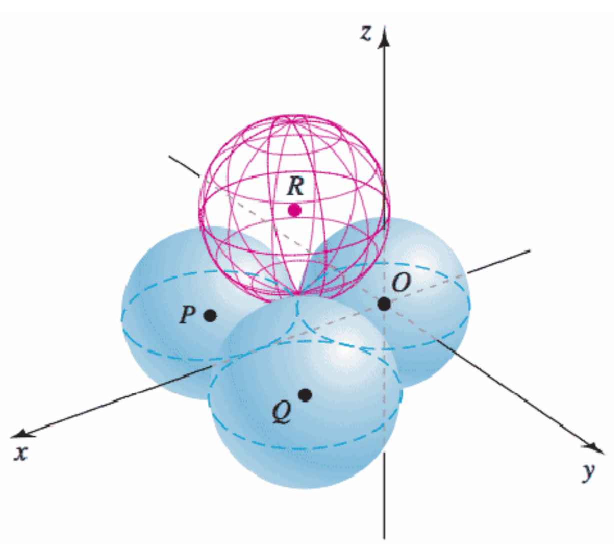


a. Find $\mathbf{r}_{0j}$ for $j = 1, 2, \ldots, 6$.

b. Find $\mathbf{r}_{12}$, $\mathbf{r}_{34}$, and $\mathbf{r}_{61}$.

c. Imagine circle 7 is added to the arrangement as shown in the figure. Find $\mathbf{r}_{07}$, $\mathbf{r}_{17}$, $\mathbf{r}_{47}$, and $\mathbf{r}_{75}$.

65. Hexagonal sphere packing Imagine three unit spheres (radius equal to 1) with centers at $O(0, 0, 0)$, $P\left(\sqrt{3}, -1, 0\right)$, and $Q\left(\sqrt{3}, 1, 0\right)$. Now place another unit sphere symmetrically on top of these spheres with its center at R (see figure).



a. Find the coordinates of R. (*Hint:* The distance between the centers of any two spheres is 2.)

b. Let $\mathbf{r}_{ij}$ be the vector from the center of sphere i to the center of sphere j. Find $\mathbf{r}_{OP}$, $\mathbf{r}_{OQ}$, $\mathbf{r}_{PQ}$, $\mathbf{r}_{OR}$, and $\mathbf{r}_{PR}$.

Additional Exercises

66-70. Properties of dot products *Let* $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$. *Let c be a scalar. Prove the following vector properties.*

66. $|\mathbf{u} \cdot \mathbf{v}| \le |\mathbf{u}|\,|\mathbf{v}|$

67. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ Commutative property

68. $c\,(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\,\mathbf{v})$ Associative property

69. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ Distributive property

70. Distributive properties

a. Show that $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = |\mathbf{u}|^2 + 2\,\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2$.

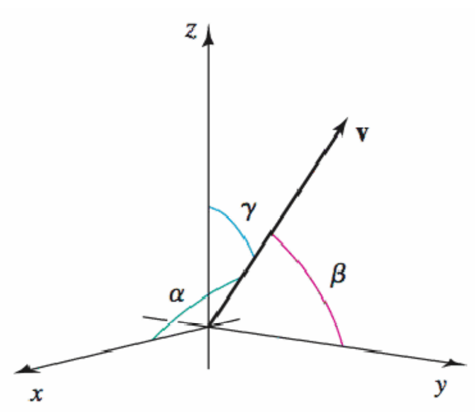
b. Show that $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = |\mathbf{u}|^2 + |\mathbf{v}|^2$ if **u** is perpendicular to **v**.

c. Show that $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = |\mathbf{u}|^2 - |\mathbf{v}|^2$.

71. Prove or disprove For fixed values of a, b, c, and d, the value of $\text{proj}_{\langle ka, kb \rangle} \langle c, d \rangle$ is constant for all nonzero values of k, for $\langle a, b \rangle \ne \langle 0, 0 \rangle$.

72. Orthogonal lines Recall that two lines $y = m\,x + b$ and $y = n\,x + c$ are orthogonal provided $m\,n = -1$ (the slopes are negative reciprocals of each other). Prove that the condition $m\,n = -1$ is equivalent to the orthogonality condition $\mathbf{u} \cdot \mathbf{v} = 0$, where **u** points in the direction of one line and **v** points in the direction of the other line.

73. Direction angles and cosines Let $\mathbf{v} = \langle a, b, c \rangle$ and let α, β, and γ be the angles between **v** and the positive x-axis, the positive y-axis, and the positive z-axis, respectively (see figure).



a. Prove that $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.

b. Find a vector that makes a 45 ° angle with **i** and **j**. What angle does it make with **k**?

c. Find a vector that makes a 60 ° angle with **i** and **j**. What angle does it make with **k**?

d. Is there a vector that makes a 30 ° angle with **i** and **j**? Explain.

e. Find a vector **v** such that $\alpha = \beta = \gamma$. What is the angle?

74-78. Cauchy-Schwarz Inequality *The definition* $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$ *implies that* $|\mathbf{u} \cdot \mathbf{v}| \le |\mathbf{u}|\,|\mathbf{v}|$ *(because* $|\cos\theta| \le 1$*). This inequality holds in any number of dimensions and has many consequences.*

74. What conditions on **u** and **v** lead to equality in the Cauchy-Schwarz Inequality?

75. Verify that the Cauchy-Schwarz Inequality holds for $\mathbf{u} = \langle 3, -5, 6 \rangle$ and $\mathbf{v} = \langle -8, 3, 1 \rangle$.

76. Geometric-arithmetic mean Use the vectors $\mathbf{u} = \left\langle \sqrt{a}, \sqrt{b} \right\rangle$ and $\mathbf{v} = \left\langle \sqrt{b}, \sqrt{a} \right\rangle$ to show that $\sqrt{a\,b} \le (a+b)/2$, where $a \ge 0$ and $b \ge 0$.

77. Triangle Inequality Consider the vectors **u**, **v**, and **u** + **v** (in any number of dimensions). Use the following steps to prove that $|\mathbf{u}+\mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$.

a. Show that $|\mathbf{u}+\mathbf{v}|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = |\mathbf{u}|^2 + 2\,\mathbf{u}\cdot\mathbf{v} + |\mathbf{v}|^2$.

b. Use the Cauchy-Schwarz Inequality to show that $|\mathbf{u}+\mathbf{v}|^2 \le (|\mathbf{u}| + |\mathbf{v}|)^2$.

c. Conclude that $|\mathbf{u}+\mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$.

d. Interpret the Triangle Inequality geometrically in $\mathbb{R}^2$ or $\mathbb{R}^3$.

78. Algebra inequality Show that for real numbers u_1, u_2, and u_3, it is true that

$$(u_1 + u_2 + u_3)^2 \le 3\left(u_1^2 + u_2^2 + u_3^2\right).$$

Use the Cauchy-Schwarz Inequality in three dimensions with $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and choose **v** in the right way.

79. Diagonals of a parallelogram Consider the parallelogram with adjacent sides **u** and **v**.

a. Show that the diagonals of the parallelogram are **u** + **v** and **u** − **v**.

b. Prove that the diagonals have the same length if and only if $\mathbf{u}\cdot\mathbf{v} = 0$.

c. Show that the sum of the squares of the lengths of the diagonals equals the sum of the squares of the lengths of the sides.

80. Distance between a point and a line in the plane Use projections to find a general formula for the distance between the point $P(x_0,\ y_0)$ and the line $a\,x + b\,y = c$. (See Exercises 52-55.)