11.4 Cross Products

The dot product combines two vectors to produce a *scalar* result. There is an equally fundamental way to combine two vectors in $\mathbb{R}^3$ and obtain a *vector* result. This operation, known as the *cross product* (or *vector product*) may be motivated by a physical application.

Suppose you want to loosen a bolt with a wrench. As you apply force to the end of the wrench in the plane perpendicular to the bolt, the "twisting power" you generate depends on three variables:

- the magnitude of the force **F** applied to the wrench;
- the length $|\mathbf{r}|$ of the wrench;
- the angle at which the force is applied to the wrench.

The twisting generated by a force acting at a distance from a pivot point is called **torque** (from the Latin *to twist*). The torque is a vector whose magnitude is proportional to $|\mathbf{F}|$, $|\mathbf{r}|$, and $\sin\theta$, where θ is the angle between **F** and **r** (Figure 11.55). If the force is applied parallel to the wrench—for example, if you pull the wrench ($\theta = 0$) or push the wrench ($\theta = \pi$)—there is no twisting effect; if the force is applied perpendicular to the wrench ($\theta = \pi/2$), the twisting effect is maximized. The direction of the torque vector is defined to be orthogonal to both **F** and **r**. As we will see shortly, the torque is expressed in terms of the cross product of **F** and **r**.

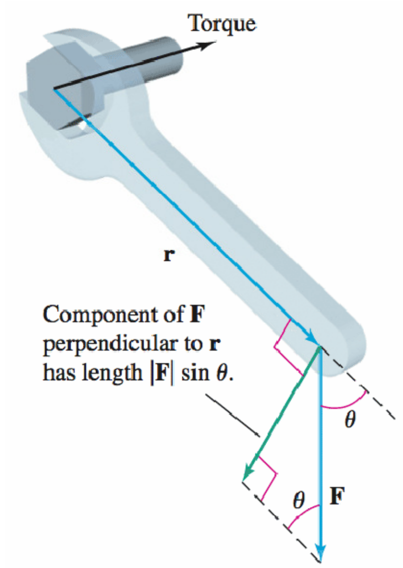


FIGURE 11.55

The Cross Product

Properties of the Cross Product

Applications of the Cross Product

Quick Quiz

SECTION 11.4 EXERCISES

Review Questions

1. Explain how to find the magnitude of the cross product $\mathbf{u} \times \mathbf{v}$.

2. Explain how to find the direction of the cross product $\mathbf{u} \times \mathbf{v}$.

3. What is the magnitude of the cross product of two parallel vectors?

4. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, what is the magnitude of $\mathbf{u} \times \mathbf{v}$?

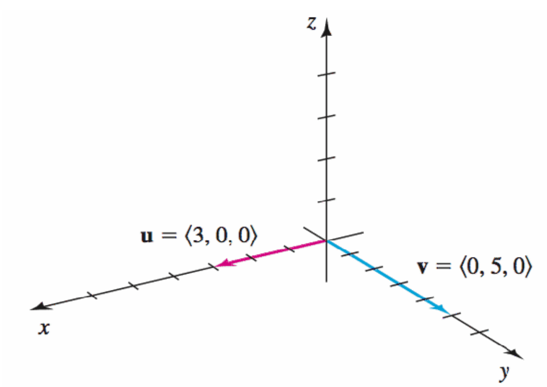
5. Explain how to use a determinant to compute $\mathbf{u} \times \mathbf{v}$.

6. Explain how to find the torque produced by a force using cross products.

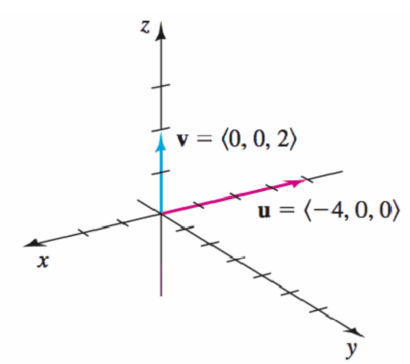
Basic Skills

7-8. Cross products from the definition *Find the magnitude of the cross product of the vectors* **u** *and* **v** *given in each figure.*

7.



8.



9-12. Cross products from the definition *Sketch the following vectors* **u** *and* **v**. *Then compute* $|\mathbf{u} \times \mathbf{v}|$ *and show the cross product on your sketch.*

9. $\mathbf{u} = \langle 0, -2, 0 \rangle$, $\mathbf{v} = \langle 0, 1, 0 \rangle$

10. $\mathbf{u} = \langle 0, 4, 0 \rangle$, $\mathbf{v} = \langle 0, 0, -8 \rangle$

11. $\mathbf{u} = \langle 3, 3, 0 \rangle$, $\mathbf{v} = \left\langle 3, 3, 3\sqrt{2} \right\rangle$

12. $\mathbf{u} = \langle 0, -2, -2 \rangle$, $\mathbf{v} = \langle 0, 2, -2 \rangle$

13-18. Coordinate unit vectors *Compute the following cross products. Then make a sketch showing the two vectors and their cross product.*

13. $\mathbf{j} \times \mathbf{k}$

14. $\mathbf{i} \times \mathbf{k}$

15. $-\mathbf{j} \times \mathbf{k}$

16. $3\,\mathbf{j} \times \mathbf{i}$

17. $-2\,\mathbf{i} \times 3\,\mathbf{k}$

18. $2\,\mathbf{j} \times (-5)\,\mathbf{i}$

19-22. Area of a parallelogram *Find the area of the parallelogram that has two adjacent sides* **u** *and* **v**.

19. $\mathbf{u} = 3\,\mathbf{i} - \mathbf{j},\ \mathbf{v} = 3\,\mathbf{j} + 2\,\mathbf{k}$

20. $\mathbf{u} = -3\,\mathbf{i} + 2\,\mathbf{k},\ \mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

21. $\mathbf{u} = 2\,\mathbf{i} - \mathbf{j} - 2\,\mathbf{k},\ \mathbf{v} = 3\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$

22. $\mathbf{u} = 8\,\mathbf{i} + 2\,\mathbf{j} - 3\,\mathbf{k},\ \mathbf{v} = 2\,\mathbf{i} + 4\,\mathbf{j} - 4\,\mathbf{k}$

23-28. Computing cross products *Find the cross products* $\mathbf{u} \times \mathbf{v}$ *and* $\mathbf{v} \times \mathbf{u}$ *for the following vectors* **u** *and* **v**.

23. $\mathbf{u} = \langle 3, 5, 0\rangle,\ \mathbf{v} = \langle 0, 3, -6\rangle$

24. $\mathbf{u} = \langle -4, 1, 1\rangle,\ \mathbf{v} = \langle 0, 1, -1\rangle$

25. $\mathbf{u} = \langle 2, 3, -9\rangle,\ \mathbf{v} = \langle -1, 1, -1\rangle$

26. $\mathbf{u} = \langle 3, -4, 6\rangle,\ \mathbf{v} = \langle 1, 2, -1\rangle$

27. $\mathbf{u} = 3\,\mathbf{i} - \mathbf{j} - 2\,\mathbf{k},\ \mathbf{v} = \mathbf{i} + 3\,\mathbf{j} - 2\,\mathbf{k}$

28. $\mathbf{u} = 2\,\mathbf{i} - 10\,\mathbf{j} + 15\,\mathbf{k},\ \mathbf{v} = 0.5\,\mathbf{i} + \mathbf{j} - 0.6\,\mathbf{k}$

29-32. Normal vectors *Find a vector normal to the given vectors.*

29. $\langle 0, 1, 2\rangle$ and $\langle -2, 0, 3\rangle$

30. $\langle 1, 2, 3\rangle$ and $\langle -2, 4, -1\rangle$

31. $\langle 8, 0, 4\rangle$ and $\langle -8, 2, 1\rangle$

32. $\langle 6, -2, 4\rangle$ and $\langle 1, 2, 3\rangle$

33-36. Computing torque *Answer the following questions about torque.*

33. Let $\mathbf{r} = \overrightarrow{OP} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. A force $\mathbf{F} = \langle 20, 0, 0\rangle$ is applied at P. Find the torque about O that is produced.

34. Let $\mathbf{r} = \overrightarrow{OP} = \mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$. A force $\mathbf{F} = \langle 10, 10, 0\rangle$ is applied at P. Find the torque about O that is produced.

35. Let $\mathbf{r} = \overrightarrow{OP} = 10\,\mathbf{i}$. Which is greater (in magnitude): the torque about O when a force $\mathbf{F} = 5\,\mathbf{i} - 5\,\mathbf{k}$ is applied at P or the torque about O when a force $\mathbf{F} = 4\,\mathbf{i} - 3\,\mathbf{j}$ is applied at P?

36. A pump handle has a pivot at (0, 0, 0) and extends to $P(5, 0, -5)$. A force $\mathbf{F} = \langle 1, 0, -10 \rangle$ is applied at P. Find the magnitude and direction of the torque about the pivot.

37-40. Force on a moving charge *Answer the following questions about force on a moving charge.*

37. A particle with unit charge ($q = 1$) enters a constant magnetic field $\mathbf{B} = \mathbf{i} + \mathbf{j}$ with a velocity $\mathbf{v} = 20\,\mathbf{k}$. Find the magnitude and direction of the force on the particle. Make a sketch of the magnetic field, the velocity, and the force.

38. A particle with unit negative charge ($q = -1$) enters a constant magnetic field $\mathbf{B} = 5\,\mathbf{k}$ with a velocity $\mathbf{v} = \mathbf{i} + 2\,\mathbf{j}$. Find the magnitude and direction of the force on the particle. Make a sketch of the magnetic field, the velocity, and the force.

39. An electron ($q = -1.6 \times 10^{-19}$ C) enters a constant 2-T magnetic field at an angle of 45° to the field with a speed of 2×10^5 m/s. Find the magnitude of the force on the electron.

40. A proton ($q = 1.6 \times 10^{-19}$ C) with velocity $2 \times 10^6\,\mathbf{j}$ m/s experiences a force $\mathbf{F} = 5 \times 10^{-12}\,\mathbf{k}$ (N) as it passes through the origin. Find the magnitude and direction of the magnetic field at that instant.

Further Explorations

41. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

- **a.** The cross product of two nonzero vectors is a nonzero vector.
- **b.** $|\mathbf{u} \times \mathbf{v}|$ is less than both $|\mathbf{u}|$ and $|\mathbf{v}|$.
- **c.** If $\mathbf{u}$ points east and $\mathbf{v}$ points south, then $\mathbf{u} \times \mathbf{v}$ points west.
- **d.** If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ and $\mathbf{u} \cdot \mathbf{v} = 0$, then either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ (or both).
- **e.** Law of Cancellation? If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

42-45. Areas of parallelograms *Find the area of the following parallelograms P.*

42. Two of the adjacent sides of P are $\mathbf{u} = \langle 4, 0, 0 \rangle$ and $\mathbf{v} = \langle 8, 8, 8 \rangle$.

43. Two of the adjacent sides of P are $\mathbf{u} = \langle -1, 1, 1 \rangle$ and $\mathbf{v} = \langle 0, -1, 1 \rangle$.

44. Three vertices of P are $O(0, 0, 0)$, $Q(4, 4, 0)$, and $R(6, 6, 3)$.

45. Three vertices of P are $O(0, 0, 0)$, $Q(2, 4, 8)$, and $R(1, 4, 10)$.

46-49. Areas of triangles *Find the area of the following triangles T. (The area of a triangle is half the area of the corresponding parallelogram.)*

46. The sides of T are $\mathbf{u} = \langle 0, 6, 0 \rangle$, $\mathbf{v} = \langle 4, 4, 4 \rangle$, and $\mathbf{u} - \mathbf{v}$.

47. The sides of T are $\mathbf{u} = \langle 3, 3, 3 \rangle$, $\mathbf{v} = \langle 6, 0, 6 \rangle$, and $\mathbf{u} - \mathbf{v}$.

48. The vertices of T are $O(0, 0, 0)$, $P(2, 4, 6)$, and $Q(3, 5, 7)$.

49. The vertices of T are $O(0, 0, 0)$, $P(1, 2, 3)$, and $Q(6, 5, 4)$.

50. A unit cross product Under what conditions is $\mathbf{u} \times \mathbf{v}$ a unit vector?

51. Vector equation Find all vectors $\mathbf{u}$ that satisfy the equation

$$\langle 1, 1, 1 \rangle \times \mathbf{u} = \langle -1, -1, 2 \rangle.$$

52. Vector equation Find all vectors $\mathbf{u}$ that satisfy the equation

$$\langle 1, 1, 1\rangle \times \mathbf{u} = \langle 0, 0, 1\rangle.$$

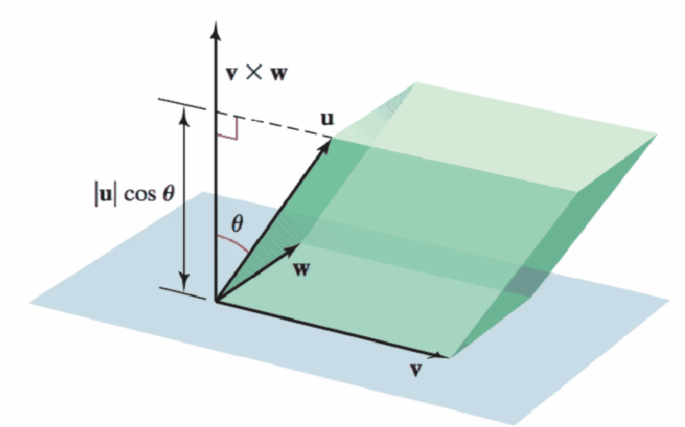
53. Area of a triangle Find the area of the triangle with vertices on the coordinate axes at the points $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$, in terms of a, b, and c.

54-56. Scalar triple product *Another operation with vectors is the* ***scalar triple product****, defined to be* $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$*, for vectors* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *in* $\mathbb{R}^3$.

54. Express $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in terms of their components and show that $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ equals the determinant

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

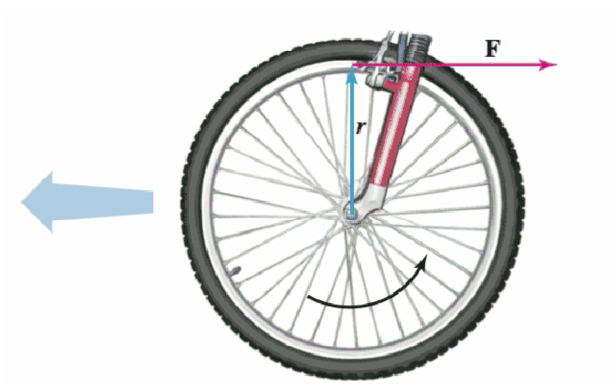
55. Consider the parallelepiped (slanted box) determined by the position vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ (see figure). Show that the volume of the parallelepiped is $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$.



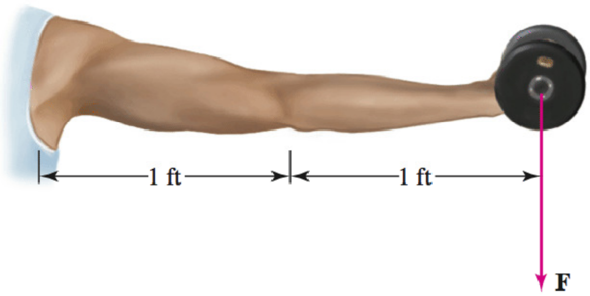
56. Prove that $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$.

Applications

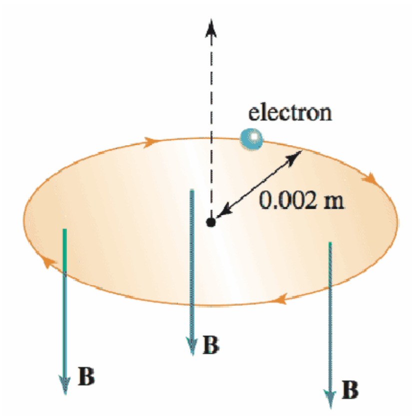
57. Bicycle brakes A set of caliper brakes exerts a force on the rim of a bicycle wheel that creates a frictional force $\mathbf{F}$ of 40 N (see figure). Assuming the wheel has a radius of 66 cm, find the magnitude and direction of the torque about the axle of the wheel.



58. Arm torque A horizontally outstretched arm supports a weight of 20 lb in a hand (see figure). If the distance from the shoulder to the elbow is 1 ft and the distance from the elbow to the hand is 1 ft, find the magnitude and describe the direction of the torque about (a) the shoulder and (b) the elbow. (The units of torque in this case are ft-lb.)



59. **Electron speed** An electron with a mass of 9.1×10^{-31} kg and a charge of -1.6×10^{-19} C travels in a circular path with no loss of energy in a magnetic field of 0.05 T that is orthogonal to the path of the electron. If the radius of the path is 0.002 m, what is the speed of the electron?



Additional Exercises

60. $\mathbf{u} \times \mathbf{u}$ Prove that $\mathbf{u} \times \mathbf{u} = \mathbf{0}$ in three ways.

- **a.** Use the definition of the cross product.
- **b.** Use the determinant formulation of the cross product.
- **c.** Use the property that $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.

61. **Associative property** Prove in two ways that for scalars a and b, $(a\,\mathbf{u}) \times (b\,\mathbf{v}) = a\,b\,(\mathbf{u} \times \mathbf{v})$. Use the definition of the cross product and the determinant formula.

62-64. Possible identities *Determine whether the following statements are true using a proof or counterexample. Assume that* **u**, **v**, *and* **w** *are nonzero vectors in* $\mathbb{R}^3$.

62. $\mathbf{u} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$

63. $(\mathbf{u} - \mathbf{v}) \times (\mathbf{u} + \mathbf{v}) = 2\,\mathbf{u} \times \mathbf{v}$

64. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$

65-66. Identities *Prove the following identities. Assume that* **u**, **v**, **w**, *and* **x** *are nonzero vectors in* $\mathbb{R}^3$.

65. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v}\,(\mathbf{u} \cdot \mathbf{w}) - \mathbf{w}\,(\mathbf{u} \cdot \mathbf{v})$, ($\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ is called the **vector triple product**.)

66. $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{x}) = (\mathbf{u} \cdot \mathbf{w})\,(\mathbf{v} \cdot \mathbf{x}) - (\mathbf{u} \cdot \mathbf{x})\,(\mathbf{v} \cdot \mathbf{w})$

67. **Cross product equations** Suppose **u** and **v** are nonzero vectors in $\mathbb{R}^3$.

a. Prove that the equation $\mathbf{u} \times \mathbf{z} = \mathbf{v}$ has a nonzero solution $\mathbf{z}$ if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. (*Hint:* Take the dot product of both sides with $\mathbf{v}$.)

b. Explain this result geometrically.