# 11.5 Lines and Curves in Space

Imagine a projectile moving along a path in three-dimensional space; it could be an electron or a comet, a soccer ball or a rocket. If you take a snapshot of the object, its position is described by a static position vector $\mathbf{r} = \langle x, y, z \rangle$. However, if you want to describe the full trajectory of the object as it unfolds in time, you must use a position vector such as $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ whose components change in time (Figure 11.66). The goal of this section is to describe continuous motion by making vectors into vector-valued functions.

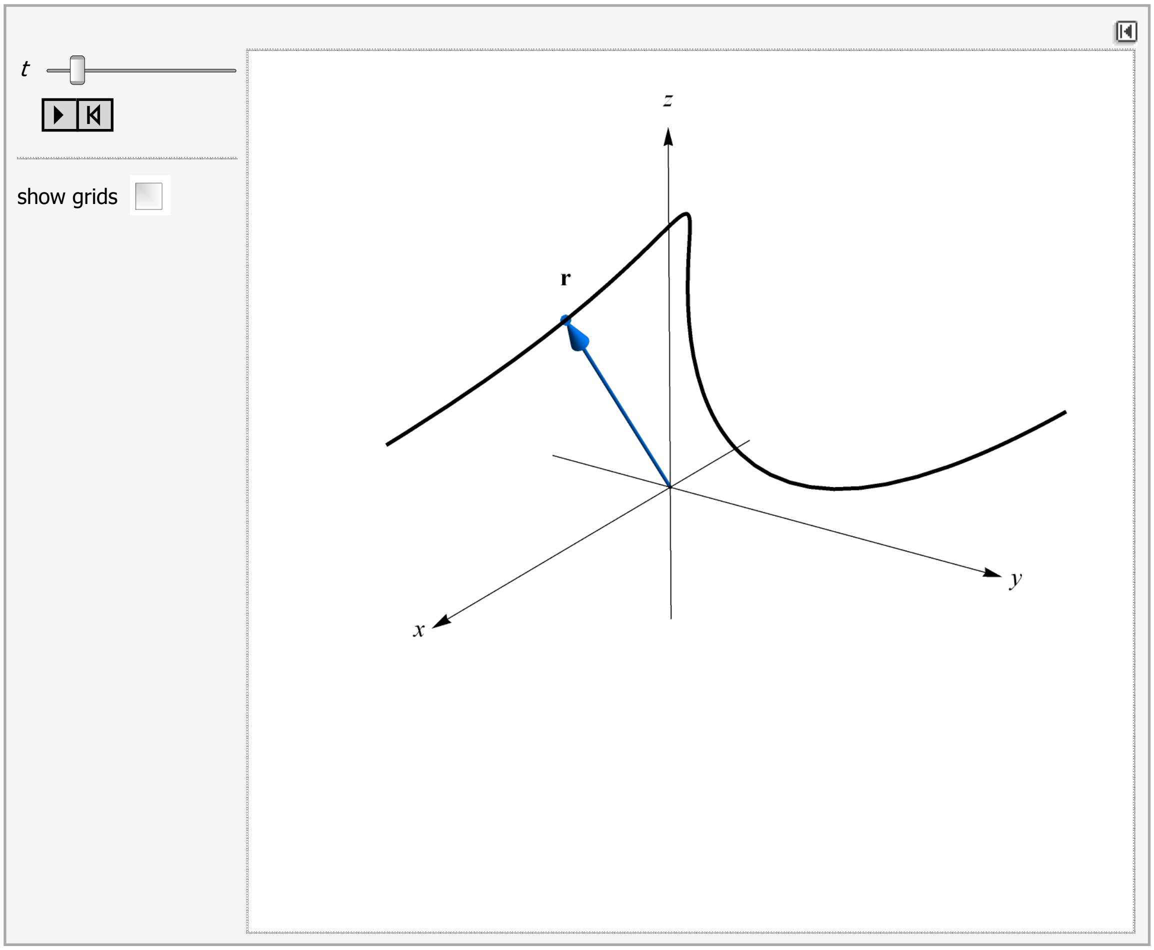



FIGURE 11.66

## Vector-Valued Functions

## Lines in Space

## Curves in Space

## Limits and Continuity for Vector-Valued Functions

## Quick Quiz

# SECTION 11.5 EXERCISES

## Review Questions

**1.** How many independent variables does the function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ have?

**2.** How many dependent scalar variables does the function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ have?

**3.** Why is $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ called a vector-valued function?

**4.** Explain how to find a vector in the direction of the line segment from $P_0(x_0, y_0, z_0)$ to $P_1(x_1, y_1, z_1)$.

**5.** How do you find an equation for the line through the points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$?

**6.** In what plane does the curve $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{k}$ lie?

**7.** How do you evaluate $\lim\limits_{t\to a}\mathbf{r}(t)$, where $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$?

**8.** How do you determine whether $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ is continuous at $t = a$?

## Basic Skills

**9-16. Equations of lines** Find an equation of the following lines. Make a sketch of the line.

**9.** The line through $(0, 0, 1)$ parallel to the $y$-axis

**10.** The line through $(0, 0, 1)$ parallel to the $x$-axis

**11.** The line through $(0, 1, 1)$ parallel to $\langle 2, -2, 2\rangle$

**12.** The line through $(0, 0, 1)$ and $(0, 1, 1)$

**13.** The line through $(0, 0, 0)$ and $(1, 2, 3)$

**14.** The line through $(1, 0, 1)$ and $(3, -3, 3)$

**15.** The line through $(-3, 4, 6)$ and $(5, -1, 0)$

**16.** The line through $(0, 4, 8)$ and $(10, -5, -4)$

**17-20. Line segments** *Find an equation of the line segment joining the given pairs of points.*

**17.** $(0, 0, 0)$ and $(1, 2, 3)$

**18.** $(1, 0, 1)$ and $(0, -2, 1)$

**19.** $(2, 4, 8)$ and $(7, 5, 3)$

**20.** $(-1, -8, 4)$ and $(-9, 5, -3)$

**21-28. Curves in space** *Graph the curves described by the following functions. Try to anticipate the shape of the curve before using a graphing utility.*

**21.** $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{k}$, for $0 \le t \le 2\pi$

**22.** $\mathbf{r}(t) = 4\cos t\,\mathbf{j} + 16\sin t\,\mathbf{k}$, for $0 \le t \le 2\pi$

**23.** $\mathbf{r}(t) = \cos \pi t\,\mathbf{i} + 2t\,\mathbf{j} + \sin \pi t\,\mathbf{k}$, for $0 \le t \le 2$

**24.** $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \sin t\,\mathbf{k}$, for $0 \le t \le 2\pi$

**25.** $\mathbf{r}(t) = 2t\,\mathbf{i} + t^2\,\mathbf{j} + 3t\,\mathbf{k}$, for $0 \le t \le 4$

**26.** $\mathbf{r}(t) = 4\sin t\,\mathbf{i} + 4\cos t\,\mathbf{j} + e^{-t}\,\mathbf{k}$, for $0 \le t < \infty$

T **27.** $\mathbf{r}(t) = e^{-t}\sin t\,\mathbf{i} + e^{-t}\cos t\,\mathbf{j} + \mathbf{k}$, for $0 \le t < \infty$

**28.** $\mathbf{r}(t) = e^{-t}\,\mathbf{i} + 3\cos t\,\mathbf{j} + 3\sin t\,\mathbf{k}$, for $0 \le t < \infty$

**29-32. Exotic curves** *Graph the curves described by the following functions. Use analysis to anticipate the shape of the curve before using a graphing utility.*

**29.** $\mathbf{r}(t) = 0.5\cos 15t\,\mathbf{i} + (8 + \sin 15t)\cos t\,\mathbf{j} + (8 + \sin 15t)\sin t\,\mathbf{k}$, for $0 \le t \le 2\pi$

**30.** $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + \cos 10t\,\mathbf{k}$, for $0 \le t \le 2\pi$

**31.** $\mathbf{r}(t) = \cos t^2\,\mathbf{i} + \sin t^2\,\mathbf{j} + t/(t+1)\,\mathbf{k}$, for $0 \le t < \infty$

**32.** $\mathbf{r}(t) = \cos t \sin 3t\,\mathbf{i} + \sin t \sin 3t\,\mathbf{j} + \sqrt{t}\,\mathbf{k}$, for $0 \le t \le 9$

**33-36. Limits** *Evaluate the following limits.*

**33.** $\displaystyle\lim_{t\to\pi/2}\left(\cos 2t\,\mathbf{i} - 4\sin t\,\mathbf{j} + \frac{2t}{\pi}\,\mathbf{k}\right)$

**34.** $\displaystyle\lim_{t\to\ln 2}\left(2e^{t}\,\mathbf{i} + 6e^{-t}\,\mathbf{j} - 4e^{-2t}\,\mathbf{k}\right)$

**35.** $\displaystyle\lim_{t\to\infty}\left(e^{-t}\,\mathbf{i} - \frac{2t}{t+1}\,\mathbf{j} + \tan^{-1} t\,\mathbf{k}\right)$

**36.** $\displaystyle\lim_{t\to 2}\left(\frac{t}{t^2+1}\,\mathbf{i} - 4e^{-t}\sin \pi t\,\mathbf{j} + \frac{1}{\sqrt{4t+1}}\,\mathbf{k}\right)$

## Further Explorations

**37. Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

**a.** The line $\mathbf{r}(t) = \langle 3, -1, 4\rangle + t\langle 6, -2, 8\rangle$ passes through the origin.

**b.** Any two nonparallel lines in $\mathbb{R}^3$ intersect.

**c.** The curve $\mathbf{r}(t) = \langle e^{-t}, \sin t, -\cos t\rangle$ approaches a circle as $t \to \infty$.

**d.** If $\mathbf{r}(t) = e^{-t^2}\langle 1, 1, 1\rangle$ then $\displaystyle\lim_{t\to\infty}\mathbf{r}(t) = \lim_{t\to-\infty}\mathbf{r}(t)$.

**38-41. Domains** *Find the domain of the following vector-valued functions.*

**38.** $\mathbf{r}(t) = \dfrac{2}{t-1}\,\mathbf{i} + \dfrac{3}{t+2}\,\mathbf{j}$

**39.** $\mathbf{r}(t) = \sqrt{t+2}\,\mathbf{i} + \sqrt{2-t}\,\mathbf{j}$

**40.** $\mathbf{r}(t) = \cos 2t\,\mathbf{i} + e^{\sqrt{t}}\,\mathbf{j} + \frac{12}{t}\,\mathbf{k}$

**41.** $\mathbf{r}(t) = \sqrt{4 - t^2}\,\mathbf{i} + \sqrt{t}\,\mathbf{j} - \frac{2}{\sqrt{1+t}}\,\mathbf{k}$

**42-45. Line-plane intersections** *Find the point (if it exists) at which the following planes and lines intersect.*

**42.** $x = 3$; $\mathbf{r}(t) = \langle t, t, t \rangle$, for $-\infty < t < \infty$

**43.** $z = 4$; $\mathbf{r}(t) = \langle 2t + 1, -t + 4, t - 6 \rangle$, for $-\infty < t < \infty$

**44.** $y = -2$; $\mathbf{r}(t) = \langle 2t + 1, -t + 4, t - 6 \rangle$, for $-\infty < t < \infty$

**45.** $z = -8$; $\mathbf{r}(t) = \langle 3t - 2, t - 6, -2t + 4 \rangle$, for $-\infty < t < \infty$

**46-48. Curve-plane intersections** *Find the points (if they exist) at which the following planes and curves intersect.*

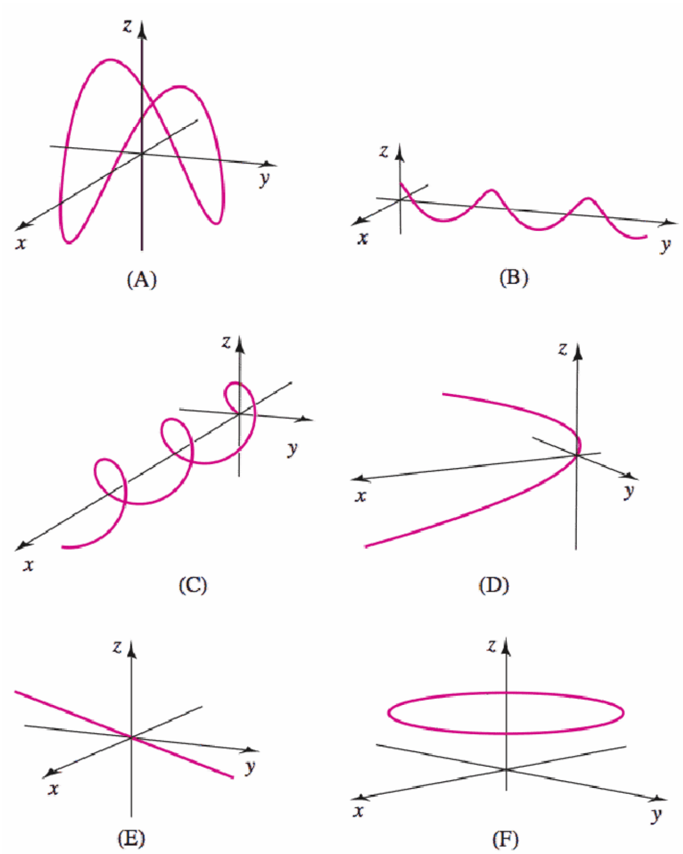
**46.** $y = 1$; $\mathbf{r}(t) = \langle 10\cos t, 2\sin t, 1 \rangle$, for $0 \le t \le 2\pi$

**47.** $z = 16$; $\mathbf{r}(t) = \langle t, 2t, 4 + 3t \rangle$, for $-\infty < t < \infty$

**48.** $y + x = 0$; $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$, for $0 \le t \le 4\pi$

**49. Matching functions with graphs** Match functions a-f with the appropriate graphs A-F.

- **a.** $\mathbf{r}(t) = \langle t, -t, t \rangle$
- **b.** $\mathbf{r}(t) = \langle t^2, t, t \rangle$
- **c.** $\mathbf{r}(t) = \langle 4\cos t, 4\sin t, 2 \rangle$
- **d.** $\mathbf{r}(t) = \langle 2t, \sin t, \cos t \rangle$
- **e.** $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle$
- **f.** $\mathbf{r}(t) = \langle \sin t, 2t, \cos t \rangle$




**50. Intersecting lines and colliding particles** Consider the lines

$$\mathbf{r}(t) = \langle 2 + 2t,\ 8 + t,\ 10 + 3t \rangle, \quad \text{for } -\infty < t < \infty$$
$$\mathbf{R}(s) = \langle 6 + s,\ 10 - 2s,\ 16 - s \rangle, \quad \text{for } -\infty < s < \infty.$$

**a.** Determine whether the lines intersect (have a common point) and if so, find the coordinates of that point.

**b.** If $\mathbf{r}$ and $\mathbf{R}$ describe the paths of two particles, do the particles collide? Assume $t \geq 0$ and $s \geq 0$ measure time in seconds.

**51. Upward path** Consider the curve described by the vector function $\mathbf{r}(t) = \left(50e^{-t}\cos t\right)\mathbf{i} + \left(50e^{-t}\sin t\right)\mathbf{j} + \left(5 - 5e^{-t}\right)\mathbf{k}$, for $t \geq 0$.

**a.** What is the initial point of the path corresponding to $\mathbf{r}(0)$?

**b.** What is $\lim_{t\to\infty} \mathbf{r}(t)$?

**c.** Sketch the curve.

**d.** Eliminate the parameter $t$ to show that $z = 5 - r/10$, where $r^2 = x^2 + y^2$.

**52-55. Closed plane curves** *Consider the curve* $\mathbf{r}(t) = (a\cos t + b\sin t)\,\mathbf{i} + (c\cos t + d\sin t)\,\mathbf{j} + (e\cos t + f\sin t)\,\mathbf{k}$ *where a, b, c, d, e, and f are real numbers. It can be shown that this curve lies in a plane.*

**52.** Assuming the curve lies in a plane, show that it is a circle centered at the origin with radius $R$ provided $a^2 + c^2 + e^2 = b^2 + d^2 + f^2 = R^2$ and $ab + cd + ef = 0$.

T **53.** Graph the following curve and describe it in words.

$$\mathbf{r}(t) = \left(\frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{3}}\sin t\right)\mathbf{i} + \left(-\frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{3}}\sin t\right)\mathbf{j} + \left(\frac{1}{\sqrt{3}}\sin t\right)\mathbf{k}$$

T **54.** Graph the following curve and describe it in words.

$$\mathbf{r}(t) = (2\cos t + 2\sin t)\,\mathbf{i} + (-\cos t + 2\sin t)\,\mathbf{j} + (\cos t - 2\sin t)\,\mathbf{k}$$

**55.** Find a general expression for a nonzero vector orthogonal to the plane containing the curve

$$\mathbf{r}(t) = (a\cos t + b\sin t)\,\mathbf{i} + (c\cos t + d\sin t)\,\mathbf{j} + (e\cos t + f\sin t)\,\mathbf{k},$$

where $\langle a, c, e\rangle \times \langle b, d, f\rangle \neq \mathbf{0}$.

## Applications

*Applications of parametric curves are considered in detail in Section 11.7.*

T **56. Golf slice** A golfer launches a tee shot down a horizontal fairway and it follows a path given by $\mathbf{r}(t) = \langle a\,t,\ (75 - 0.1\,a)\,t,\ -5\,t^2 + 80\,t\rangle$, where $t \geq 0$ measures time in seconds and $\mathbf{r}$ has units of feet. The $y$-axis points straight down the fairway and the $z$-axis points vertically upward. The parameter $a$ is the slice factor that determines how much the shot deviates from a straight path down the fairway.

**a.** With no slice ($a = 0$), sketch and describe the shot. How far does the ball travel horizontally (the distance between the point the ball leaves the ground and the point where it first strikes the ground)?

**b.** With a slice ($a = 0.2$), sketch and describe the shot. How far does the ball travel horizontally?

**c.** How far does the ball travel horizontally with $a = 2.5$?

## Additional Exercises

**57-59. Curves on spheres**

T **57.** Graph the curve $\mathbf{r}(t) = \left\langle \frac{1}{2}\sin 2\,t,\ \frac{1}{2}(1 - \cos 2\,t),\ \cos t\right\rangle$ and prove that it lies on the surface of a sphere centered at the origin.

**58.** Prove that for integers $m$ and $n$, the curve

$$\mathbf{r}(t) = \langle a\sin m\,t\cos n\,t,\ \ b\sin m\,t\sin n\,t,\ \ c\cos m\,t\rangle$$

lies on the surface of a sphere where $a$, $b$, and $c$ are nonzero constants with $a^2 = b^2 = c^2$.

**59.** Find the period of the function in Exercise 58; that is find the smallest positive real number $T$ such that $\mathbf{r}(t + T) = \mathbf{r}(t)$ for all $t$.

**60. Limits of vector functions** Let $\mathbf{r}(t) = \langle f(t),\ g(t),\ h(t)\rangle$.

**a.** Assume that $\lim_{t\to a}\mathbf{r}(t) = \mathbf{L} = \langle L_1, L_2, L_3\rangle$, which means that $\lim_{t\to a}|\mathbf{r}(t) - \mathbf{L}| = 0$. Prove that

$$\lim_{t\to a} f(t) = L_1, \quad \lim_{t\to a} g(t) = L_2, \quad \text{and} \quad \lim_{t\to a} h(t) = L_3.$$

**b.** Assume that $\lim_{t\to a} f(t) = L_1$, $\lim_{t\to a} g(t) = L_2$, and $\lim_{t\to a} h(t) = L_3$. Prove that $\lim_{t\to a}\mathbf{r}(t) = \mathbf{L} = \langle L_1, L_2, L_3\rangle$, which means that $\lim_{t\to a}|\mathbf{r}(t) - \mathbf{L}| = 0$.